High-dimensional expanders from Kac–Moody–Steinberg groups

by

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A thesis submitted to the

Faculty of Science

Ghent University

for the degree of

Doctor of Science: Mathematics

Ghent University

June 2026

Acknowledgements

I am very grateful to Tom De Medts for his guidance and supervision throughout my PhD. The combination of giving me the freedom to develop and finding my interests while at the same time always being open to answer my questions and give advice was a great support. Special thanks to Tom De Medts, Pierre-Emmanuel Caprace and Timothée Marquis for having the idea of combining KMS groups and high-dimensional expansion, starting the EOS project together and writing the project description which has been a valuable inspiration and guideline throughout my PhD.

I would like to express my deepest gratitude to Izhar Oppenheim for the very fruitful and enjoyable collaboration. Furthermore, I am thankful to Laura Grave de Peralta for the joint work and vivid discussions.

I also want to thank the members of my jury, Inna Capdeboscq, Bart De Bruyn, Elyasheev Leibtag, Timothée Marquis, Izhar Oppenheim, and Quang Lam Pham for taking the time to read (part of) this thesis and their valuable feedback.

Many thanks to the FWO and the F.R.S.–FNRS for their generous funding under the Excellence of Science (EOS) program (project ID 40007542).

My sincere thanks go to all my colleagues for the refreshing lunch and coffee breaks, the board game nights, and for making the daily work environment so much brighter. I am grateful to Jari for patiently answering all my questions during my first years, for proofreading parts of this thesis and for the LaTeX template. I also want to thank Sam and later also Alisa for their invaluable assistance with administrative tasks, which helped keep things running smoothly.

To my family, thank you for being the reliable foundation of this journey. And finally, I want to deeply thank Lukas for his careful proofreading of this thesis and for being such a stupendous partner in life.

Abstract

Expander graphs are sparse and at the same time highly connected graphs. These properties seem contradictory, but the existence of expander graphs can be shown by a straightforward probabilistic argument, and by now many explicit constructions are known. In particular, expander graphs have proven highly useful in a variety of applications in computer science and mathematics. Thus in the early 2000s, the expander community started to generalize what it means to be highly connected to simplicial complexes as higher dimensional analogues of graphs. Unfortunately, there is not one canonical way to extend the notion of expansion, since several, for graphs equivalent, characterizations lead to different versions of high-dimensional expansion. In this work, we focus on local spectral, coboundary and cosystolic expansion. High-dimensional expanders have already proven useful in a variety of applications, for example in the construction of good property testers and quantum LDPC codes, in questions around Markov chain mixing, and they appear in considerations on the question of the existence of non-sofic groups. In contrast to the graph case, only very few constructions of high-dimensional expanders are known.

In this thesis, we construct a new family of complexes, called KMS complexes, which provides new examples of local spectral, coboundary and cosystolic expanders. The construction is based on the rich structure of Kac–Moody–Steinberg (KMS) groups, which are amalgamated products of maximal unipotent subgroups of Chevalley groups. In many cases, the KMS group is isomorphic to the maximal unipotent subgroup of the corresponding minimal Kac–Moody group. In particular, KMS groups admit many finite quotients. For each sufficiently large such quotient, we construct a simplicial complex as coset complex over the quotient. The KMS complexes have the property that the local structure only depends on the KMS group and not on the finite quotient. The local structure, the proper links, are each isomorphic to the complex opposite a chamber in a spherical building. We use the rich structure these opposite complexes inherit from the buildings to show local spectral expansion for all KMS complexes. Furthermore, we show that the opposite complexes of classical (i.e. type A_n, C_n and D_n) buildings are coboundary expanders. To achieve this, we use the machinery of cone functions and develop new methods to construct cone functions inductively. We then deduce cosystolic expansion for the classical KMS complexes. We further investigate a specific class of KMS complexes, called congruence KMS complexes, which arise from explicit quotients inside Chevalley groups. For this class, we can show vanishing of the first cohomology group and thus deduce 1-coboundary expansion for most abelian and non-abelian coefficient groups. We conclude with a discussion on how these ideas can be generalized to other groups and how KMS complexes could provide the first examples of 3-dimensional coboundary expanders.

Contents

Chapter 1

Introduction

1.1 From good networks to high-dimensional expanders

Imagine you were given the task to build a rail network completely from scratch in a country that did not have trains before. Which cities would you connect with direct rail lines? How can you make sure that not all trains stand still if one line breaks? At the same time, you have to keep the cost in mind, which should not be too high. As mathematicians, we can ignore many of the real world problems, and only focus on the task of connecting the cities, modelled as dots called vertices, by drawing straight lines, called edges, between them. We can abstract the situation even further by ignoring the precise positions of the cities on the map and just think of them as a set of labels V. The line between two cities with names v and w is then modeled by the set $\{v, w\}$. We get a set of edges E containing smaller sets with each two vertices in them. Together, the two sets $G = (V, E)$ form what we call a graph. We are now looking for graphs that model good networks, i.e. which are well connected and yet do not contain too many edges.

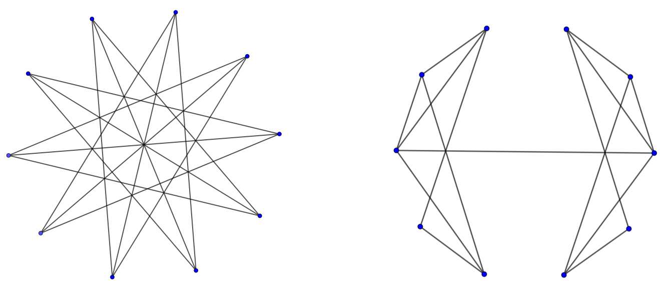


Figure 1.1: Two graphs, both with 10 vertices and 15 edges but very different connectivity properties.

Whether a graph is well connected is sometimes intuitively clear when we draw them, like in Figure 1.1. Mathematically, we have two main methods for measuring the connectivity of a graph: The Cheeger constant $h(G)$, which counts how many edges have to be cut to separate a significant part of the vertices from the rest, and the spectral gap $1 - \lambda_2(G)$, which measures how fast a random walk on the graph converges to the uniform distribution. This means that we walk on the vertices of the graph and for each step, we choose one of the neighbors of the vertex we are currently standing on and walk there next. The spectral gap then measures how many steps we have to take until the probability of being at any vertex in the graph is the same. Both terms are described in detail in Section 2.1. Fortunately, we do not have to choose one, since they are connected via the Cheeger inequality.

To enforce the sparseness, we do not only look for a single graph, but for a family of graphs $G_n, n \in \mathbb{N}$ such that

(i) the number of vertices $|V_n|$ grows to infinity,

(ii) each vertex has at most d neighbors,

(iii) for each n the graph G_n is well-connected, i.e. $h(G_n) \geq \lambda$ or equivalently $1 - \lambda_2(G_n) \geq \tilde{\lambda}$.

Here $d \in \mathbb{N}, \lambda, \tilde{\lambda} > 0$ are constants that have to be the same for all graphs in the family. If (i)-(iii) are satisfied, we call $(G_n)_{n \in \mathbb{N}}$ a family of expander graphs.

Expander graphs were first studied in the 1960s. A first proof for the existence of expander graphs was given in [Pin73] using random methods. Constructing explicit families of expanders was not trivial, but by now many constructions are known. These include group theoretic constructions using Cayley graphs [Mar73] and groups with Kaszhdan's property (T) (see e.g. [LPS88; Mar88]), but also elementary combinatorial methods like the zig-zag product [RVW00] are used to construct expander families. Expander graphs have proven very useful in a variety of applications such as the construction of superconcentrators [Kla84], good sorting networks [AKS83] and error correcting codes [HLW06].

But some problems remained open and could not be solved by considering expander graphs. As an answer to this shortcoming, the idea of generalizing the concept of expansion to higher dimension emerged.

The higher dimensional objects that we consider are so called simplicial complexes. We again have a set of vertices V, but now we allow not only for two vertices to be connected by an edge, but also for three vertices to be connected by a triangle, four by a tetrahedron etc.. Thus a simplicial complex is a collection of subsets of vertices $X \subseteq \mathcal{P}(V)$ such that

- for each vertex $v \in V$ the singleton $\{v\}$ is in X,
- for each $A \in X$ and each subset $B \subset A$ we have $B \in X$.

The first condition makes sure that our vertex set is not larger than necessary. We will slightly abuse the notation and identify v with the singleton $\{v\}$. The second condition implies that X is closed with respect to taking subsets. For example if a triangle is in X then also all the edges of the triangle have to be part of X. We call the elements of X simplices or faces. An example of the visualizations of a simplicial complex is shown in Figure 1.2. The dimension of a simplex is the number of its vertices minus one, for example the dimension of a triangle is two. The dimension of X is the maximum over the

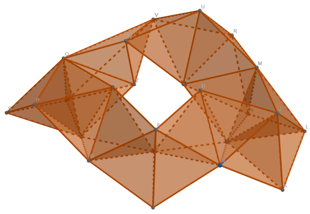


Figure 1.2: Example of a simplicial complex.

dimensions of its faces. We say that X is *pure* n-dimensional if every simplex is contained in an n-dimensional one.

We can now generalize what it means to be highly connected to simplicial complexes. There are several different approaches and unfortunately, they will turn out to not be equivalent in dimensions two and higher anymore. In all cases, we are not only interested in finding a single example (although that can already be challenging) but to construct infinite families of complexes with a uniform bound on the expansion, with growing number of vertices but with uniformly bounded degree, i.e. for each vertex the number of maximal simplices which contain it is bounded.

Spectral expansion The idea here is to generalize the spectral gap definition of expansion. We consider higher dimensional random walks as follows. We start on an i-dimensional simplex σ and sample uniformly at random an $(i+1)$-dimensional simplex τ which contains σ. We then sample an i-dimensional simplex contained in τ at random and walk to this simplex. This is called the up-walk and one can analogously define a down-walk. The convergence of these random walks is, similarly to the graph case, governed by the spectrum of a Laplace operator. We call X a λ-spectral expander if the spectral gap of the up-walk is at least λ for all $0 \leq i \leq n-1$.

A second notion, called *local spectral expansion*, generalizes the spectral expansion approach for graphs by looking at the neighborhood of each simplex, called link. If the underlying graph of each link has spectral gap at least $1-\lambda$, we call X a λ-local spectral expander. Every local spectral expander is also a spectral expander by a result of Garland which is described in [GW16a].

Coboundary and cosystolic expansion We can generalize the Cheeger constant by expressing it in terms of simplicial cohomology which then has a straightforward generalization to higher dimensions.

For a graph $G=(V,E)$ its (weighted) Cheeger constant is given by

$$h(G) = \min_{\emptyset \neq A \subsetneq V} \frac{\|E(A, V\setminus A)\|}{\min(\|A\|, \|V\setminus A\|)},$$

where $E(A, V\setminus A)$ is the set of edges with one vertex in A and the other one not in A, and given weight function $w : V \cup E \to \mathbb{R}_{>0}$ we define for a set $B \subseteq V \cup E$ its weighted size

$\|A\|= \sum_{a\in B} w(a)$. If we consider w to be the constant one function, we have $\|B\|= |B|$ and obtain the unweighted Cheeger constant. Instead of considering subsets of vertices, we can also consider functions from V to the field with two elements $\mathbb{F}_2 = \{0, 1\}$. Then $A \subseteq V$ corresponds to the function that maps all vertices in A to 1 and the others to 0. We denote the set of all functions from V to $\mathbb{F}_2$ by $C^0(G, \mathbb{F}_2)$. For each such function f, we define its boundary $d_0 f : E \to \mathbb{F}_2$ by setting $d_0 f(\{v, w\}) = f(v) + f(w)$. Note that if f is the function associated to A as above, then $d_0 f(\{v, w\}) = 1$ if and only if $\{v, w\} \in E(A, V \setminus A)$. We furthermore set $C^{-1}(G, \mathbb{F}_2) = \{f : \{\emptyset\} \to \mathbb{F}_2\}$ and for such f set $d_{-1} f(v) = f(\emptyset)$ for all $v \in V$. Hence the image of $d_{-1} : C^{-1}(G, \mathbb{F}_2) \to C^0(G, \mathbb{F}_2)$, called $B^0(G, \mathbb{F}_2)$ consists of the constant zero and constant one function on the vertices, which correspond to $\emptyset, V$ respectively. Thus we can rewrite $h(G)$ as

$$\min_{f\in C^0(G,\mathbb{F}_2)\setminus B^0(G,\mathbb{F}_2)} \frac{\|d_0 f\|}{\min_{b\in B^0(G,\mathbb{F}_2)}(\|f + b\|)}.$$

Here $\|f\|$ is the (weighted) size of the support of f, i.e. of those vertices/edges for which f has value 1.

For a higher dimensional simplicial complex X we can define $C^i(X, \mathbb{F}_2)$ as set of functions from the i-dimensional simplices of X to $\mathbb{F}_2$. The so-called coboundary operator $d_i : C^i(X, \mathbb{F}_2) \to C^{i+1}(X, \mathbb{F}_2)$ generalizes d_0 as well and we set $B^i(X, \mathbb{F}_2) = \operatorname{Im} d_{i-1}$. Then the i-coboundary expansion constant of X is defined as

$$h^i_{\mathrm{cb}}(X, \mathbb{F}_2) = \min_{f\in C^i(X,\mathbb{A})\setminus B^i(X,\mathbb{F}_2)} \frac{\|d_i f\|}{\min_{b\in B^i(X,\mathbb{F}_2)}(\|f - b\|)}.$$

Note that the norm here is not just the size of the support, but we weight the simplices in the support according to the so called Garland weights, which take the higher dimensional structure of X into account.

By taking the minimum over all f which are not in the kernel of d_i, we get a potentially larger constant called the cosystolic expansion constant, denoted $h^i_{\mathrm{cs}}(X, \mathbb{F}_2)$.

We can define the coboundary and cosystolic expansion constant not just for $\mathbb{F}_2$ but for arbitrary abelian groups, using cohomology theory. Furthermore, we can define $h^1_{\mathrm{cb}}, h^1_{\mathrm{cs}}$ also for arbitrary, potentially non-abelian groups.

Definition 1.1.1. Let X be a finite, pure n-dimensional complex, $\mathbb{A}$ an abelian group, $1 \leq k \leq n - 1$, and $\lambda, \mu > 0$. We call X a $(\lambda, \mathbb{A})$ k-coboundary expander if

$$h^i_{\mathrm{cb}}(X, \mathbb{A}) \geq \lambda \text{ for all } 0 \leq i \leq k.$$

We omit the k in the name if $k = n - 1$.

We call X a $(\lambda, \mu, \mathbb{A})$ k-cosystolic expander if

$$h^i_{\mathrm{cs}}(X, \mathbb{A}) \geq \lambda \text{ and } \|f\|\geq \mu \text{ for all } f \in \ker d_i \setminus \operatorname{Im} d_{i-1} \text{ for all } 0 \leq i \leq k.$$

Let Λ be an arbitrary group. Then we call X a (λ, Λ) 1-coboundary expander if

$$h^1_{\mathrm{cb}}(X, \Lambda) \geq \lambda.$$

In contrast to the graph case, spectral and coboundary/cosystolic expansion are no longer equivalent. For example, it was shown in [GW16b; SKM14] that the coboundary expansion constant with $\mathbb{F}_2$ coefficients and the higher-dimensional spectral gap are not comparable.

Despite the idea of high-dimensional expansion being quite new, it already sparked a variety of applications. Local spectral expanders are used to construct agreement tests in the high soundness regime [DK17; DH24; DD19]. They also lead to solving a longstanding conjecture regarding convergence of certain Markov chains by analyzing the global random walk through local analysis at the links [Ana+24]. Cosystolic expanders which are not coboundary expanders can be used to construct good quantum low density parity check codes [EKZ24] and there is further development in this direction [GK25]. Cosystolic and coboundary expansion with coefficients in the symmetric groups are used to construct agreement tests in the low soundness regime which lead to good probabilistically checkable proofs [OS25; DD24a; DD24b; DDL24]. The notion of swap cosystolic expansion, where for some k the k-dimensional simplices are now considered as vertices of a new complex, with adjacency as induced from the original complex, also plays a role here [DD24c].

Furthermore, cosystolic expansion both with $\mathbb{F}_2$ and symmetric coefficients appeared in considerations on the search of non-sofic groups [GT24; CL25a]. Soficity is a group property with some useful applications, e.g. Kaplanzky's direct finiteness conjecture holds for sofic groups [ES04]. It is conjectured by Gromov, who also introduced the term, that all groups are sofic [Gro99].

In contrast to expander graphs, not many constructions of high-dimensional expanders are known. All constructions rely on group theory. One way of constructing a simplicial complex from a group and a family of subgroups is the coset complex. It is defined as follows.

Definition 1.1.2. Given a group G and a family of subgroups $\mathcal{H} = (H_0, \dots, H_n)$ of G, we define the associated *coset complex* $\mathcal{CC}(G; (H_0, \dots, H_n))$ as the simplicial complex with vertex set $\bigsqcup_{i=0}^{n} G/H_i$ and where a set of vertices $\{g_1 H_{i_1}, \dots, g_k H_{i_k}\}$ forms a $(k-1)$-simplex in $\mathcal{CC}(G; \mathcal{H})$ if and only if their intersection is non-empty and the indices i_j are pairwise distinct.

Prior to this work, there were two main families of high-dimensional expanders, the Ramanujan complexes [Li04; LSV05] and the Kaufman–Oppenheim complexes [KO23; OP22; DLZ23]. Both families of examples are discussed in more detail in Section 2.5. The Ramanujan complexes are quotients of Bruhat–Tits buildings by certain cocompact lattices. The Kaufman–Oppenheim complexes are more elementary and can be described as coset complexes of certain matrix groups.

But both constructions have their downside. The Ramanujan complexes are spectral and cosystolic expanders, but were shown to not be coboundary expanders. Furthermore, they are not very symmetric and difficult to work with. On the other hand, the Kaufman–Oppenheim complexes are highly symmetric and easy to describe, but they are only spectral expanders in general, and only give rise to 2-dimensional coboundary expanders.

Thus, the work in this thesis is motivated by the goal of constructing new high-dimensional expanders with the novel idea of using Kac–Moody–Steinberg (KMS) groups. This lead to the development of KMS complexes, which seem to combine the best of the two previously known constructions.

KMS complexes have already been used for first applications and the construction of codes with good parameters in [Dik+26].

1.2 KMS groups and complexes

KMS groups have their origin in Kac–Moody theory, which is a natural generalization to infinite dimensions of the classical setting of semisimple Lie algebras and groups. The Kac–Moody algebras and groups are described by their generalized Cartan matrix (GCM) which can be visualized by its Dynkin diagram. In this work, we will only consider so called 2-spherical GCMs, which implies that the Dynkin diagram is a finite graph, where vertices can be connected by zero, one, two or three edges, and in the later two cases, the edges are equipped with an arrow. An example is shown in Figure 1.3. To each GCM/Dynkin

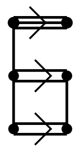


Figure 1.3: An example of a Dynkin diagram.

diagram A and field K, we can associate a group $G_A(K)$ called the minimal Kac–Moody group of type A over K. It is defined by a presentation which is shown in Definition 3.4.4.

To A and K, we can also associate the corresponding Kac–Moody–Steinberg (KMS) group. The idea is that we only want to consider the part of the Kac–Moody group governed by the so called Steinberg relation, hence the name.

Let $I = \{0, \ldots, n\}$ be the set of vertices of the Dynkin diagram A. For each $a \in I$ we let $U_a = (K, +)$ denote a copy of the additive group of the underlying field. For $a, b \in I$ we define groups U_{ab} depending on how a, b are connected in the Dynkin diagram.

(i) If a, b are not connected by an edge, then $U_{ab} = U_a \times U_b$.

(ii) If a, b are connected by a single edge, then

$$U_{ab} = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in K \right\}$$

We see $U_a \leq U_{ab}$ by identifying it matrices of the above form with $y = z = 0$, and $U_b \leq U_{ab}$ by identifying it matrices of the above form with $x = y = 0$.

(iii) If a, b are connected by a double edge with an arrow pointing to a then

$$U_{ab} = \left\{ \begin{pmatrix} 1 & w & wy+z & y+wx \\ 0 & 1 & y & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -w & 1 \end{pmatrix} \mid w, x, y, z \in K \right\} \leq \mathrm{Sp}_4(K)$$

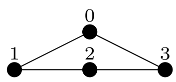
where $\mathrm{Sp}_4(K) = \{A \in \mathrm{GL}_4(K) \mid A^T J A = J\}$ for $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. We can see $U_a \leq U_{ab}$ via the entry named w above and $U_b \leq U_{ab}$ via the entry named x.

(iv) If a, b are connected by a triple edge, then U_{ab} is a group of type G_2. We will not go into details here since it is not relevant for most of the examples.

Note that if we would drop the 2-sphericiallity assumption, the groups U_{ab} would no longer be subgroups of a Chevalley group. In particular, it will later be essential that over a finite field K the groups U_{ab} are finite which requires 2-sphericiallity.

Definition 1.2.1. Let A be a 2-spherical Dynkin diagram and K a field. The KMS group $\mathcal{U}_A(K)$ of type A over K is the free product of the $U_{ab}, a, b \in I$ amalgamated along the U_a's.

The free product with amalgamation is defined in Definition 3.5.17. Intuitively, it is the free product of the $U_{a,b}$'s glued together such that the image of U_a in U_{ab} and U_{ac} coincide. It is true, although non-trivial, that the groups U_a, U_{ab} embed into $\mathcal{U}_A(K)$. We will denote their images in $\mathcal{U}_A(K)$ by U_a, U_{ab} again.

Example 1.2.2. Let A by the Dynkin diagram 

and K be a field of size at least 5. Then

$$\mathcal{U}_A(K) \cong \langle \begin{pmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \eta \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma t & 0 & 0 & 1 \end{pmatrix} \mid \lambda, \mu, \eta, \gamma \in K \rangle$$
$$\leq \mathrm{SL}_4(K[t]).$$

We have that

$$U_0 \cong \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma t & 0 & 0 & 1 \end{pmatrix} \mid \gamma \in K \rangle, \ U_1 \cong \langle \begin{pmatrix} 1 & \lambda & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \lambda \in K \rangle,$$
$$U_2 \cong \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \mu \in K \rangle, \ U_3 \cong \langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \eta \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid \eta \in K \rangle.$$

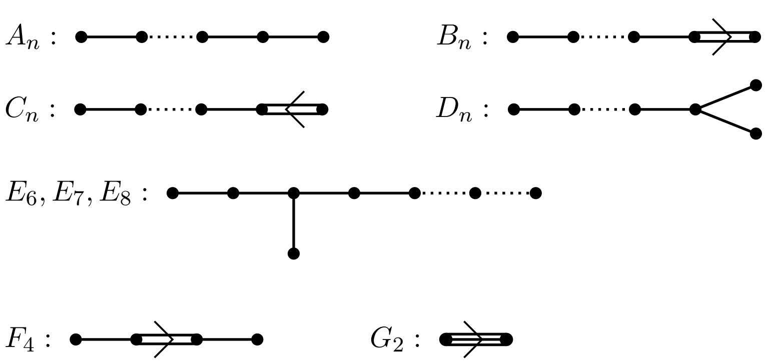


Figure 1.4: List of the connected spherical Dynkin diagrams.

Note that each two of these groups, whose corresponding vertices are connected by an edge in Dynkin diagram, generate a subgroup isomorphic to

$$\left\{ \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 1 & x_4 & x_5 \\ 0 & 0 & 1 & x_6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid x_i \in K \right\}$$

as required by (ii). To injectively embed all of them into SL_4 we need to add the variable t and consider $\mathrm{SL}_4(K[t])$.

If K is a finite field, the KMS group $\mathcal{U}_A(K)$ is finite if and only if A is a spherical Dynkin diagram. These are specific diagrams that have been classified and can be seen in Figure 1.4.

We call a subset of vertices $J \subseteq I$ spherical if the induced subdiagram is spherical, i.e. is in the list of diagrams in Figure 1.4. We call a Dynkin diagram A m-spherical, if each subdiagram on m vertices is spherical. Let $Q_A = \{J \subseteq I \mid J \text{ is spherical}\}$. For example, the diagram in Figure 1.3 is 2-spherical but not 3-spherical.

An example of a Kac–Moody–Steinberg group was first mentioned in [Ers08] as a Golod-Shafarevich group with property (T). KMS groups on Dynkin diagrams with three vertices (i.e. with rank 3) were further investigated in [Cap+22]. There is a dichotomy between KMS groups which are only 2-spherical and those which are at least 3-spherical. The latter are isomorphic to the maximal unipotent subgroup U^+ of the Kac–Moody group, while the 2-spherical KMS groups only surject onto this group. The kernel of this morphism is further investigated in [CM26].

We can now define the main object of this work, the KMS complexes.

Definition 1.2.3. Let A be a Dynkin diagram on the vertex set $I = \{0, \dots, n\}$ which is n-spherical (i.e. each proper subdiagram is spherical). Let K be a finite field with at least four elements. For each spherical $J \in Q_A$ we define its local group

$$U_J = \langle U_a \mid a \in J \rangle \leq \mathcal{U}_A(K).$$

Furthermore, let $\phi : \mathcal{U}_A(K) \to G$ be a finite quotient of $\mathcal{U}_A(K)$ such that

(i) $\phi|_{U_J}$ is injective for all $J \in Q_A$,

(ii) $\phi(U_J \cap U_L) = \phi(U_J) \cap \phi(U_L)$ for all $J, L \in Q_A$.

Then we set

$$X = \mathcal{CC}(G; (\phi(U_{I\setminus\{i\}})_{i\in I}))$$

and call X a KMS complex of type A over K.

Do finite quotients of the KMS groups satisfying the assumptions of Definition 1.2.3 exist? The straightforward answer is yes. If A is at least 3-spherical and $|K|\geq 5$ then $\mathcal{U}_A(K)$ is residually finite, and hence we can find an infinite family of finite quotients which is injective enough. If A is of rank 3 and 2-spherical, then $\mathcal{U}_A(K)$ is most likely not residually finite (it is known for $K = \mathbb{Q}$ and expected for other fields, see [CM26]), but surjects onto a group which is, and this surjection is injective enough to still give rise to finite quotients satisfying the assumptions of Definition 1.2.3. This is discussed in detail in Section 4.2.

For KMS groups of affine untwisted type (see Figure 3.4), we can construct KMS complexes explicitly in the Chevalley groups of the corresponding spherical type. These are matrix groups, for example $\mathrm{SL}_{n+1}(K)$ is the Chevalley group of type A_n over K, see Section 3.2 for more details. We present the idea with the following example which already shows all the key steps. KMS complexes of this form are called *congruence KMS complexes* and are defined in detail in Section 4.3.

Example 1.2.4. While the KMS groups $\mathcal{U}_A(K)$ is not known to be isomorphic to a subgroup of $\mathrm{SL}_3(K[t])$ (only for SL_n with $n \geq 4$) we can still construct KMS complexes using SL_3. Let $\phi : \mathcal{U}_A(K) \to \mathrm{SL}_3(K[t])$ denote the map similar to the one from Example 1.2.2 (see Section 4.3 for more details). We get the following images of the local groups of the spherical subsets of $I = \{0, 1, 2\}$:

$$\phi(U_{12}) = \begin{pmatrix} 1 & K & K \\ 0 & 1 & K \\ 0 & 0 & 1 \end{pmatrix}, \qquad \phi(U_{01}) = \begin{pmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ Kt & Kt & 1 \end{pmatrix}, \qquad \phi(U_{02}) = \begin{pmatrix} 1 & 0 & 0 \\ Kt & 1 & K \\ Kt & 0 & 1 \end{pmatrix}.$$

Here (and also below) we use a shortened notation, by which we for example mean

$$\begin{pmatrix} 1 & 0 & 0 \\ Kt & 1 & K \\ Kt & 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \lambda_1 t & 1 & \lambda_2 \\ \lambda_3 t & 0 & 1 \end{pmatrix} \mid \lambda_i \in K \right\}.$$

To construct an infinite family of KMS complexes, we fix a family of irreducible polynomials $(f_m)_{m\in\mathbb{N}}$ in $K[t]$ with increasing degrees. This gives rise to maps

$$\phi_m = \pi_{f_m} \circ \phi : \mathcal{U}_A(K) \to \mathrm{SL}_3(K[t]/(f_m))$$

by taking each entry of the matrices modulo the ideal generated by f_m.

Setting

$$G_m = \mathrm{SL}_3(K[t]/(f_m)),\ H_0^m = \pi_{f_m}(\phi(U_{12})),\ H_1^m = \pi_{f_m}(\phi(U_{02})),\ H_2^m = \pi_{f_m}(\phi(U_{01}))$$

we get an infinite family of KMS complexes of the form $X_m = \mathcal{CC}(G_m; H_0^m, H_1^m, H_2^m), m \in \mathbb{N}$.

We will use the notation $X_m = X(A_2, K, f_m)$ which denotes that we take the congruence KMS complex over the Chevalley group of type A_2 (which is SL_3) over $K[t]/(f_m)$ as the finite quotient of the KMS group. This construction works analogously for all other spherical types.

1.3 KMS complexes are high-dimensional expanders

We are now ready to state the main results of this work. Summarized in one sentence, we show that KMS complexes are good high-dimensional expanders. In more detail, we have the following.

Main Theorem A (Theorem 4.4.3)**.** Let A be a Dynkin diagram with $n+1 \geq 3$ vertices which is non-spherical but n-spherical. Let $K = \mathbb{F}_q$ be a finite field of size at least 4. We define the parameter γ depending on A and K as follows:

- If A has at most single edges, we set $\gamma = \frac{1}{\sqrt{q}}$;
- if A has at most double edges, we set $\gamma = \sqrt{\frac{2}{q}}$;
- if A has a triple edge, we set $\gamma = \sqrt{\frac{3}{q}}$.

If $\gamma < \frac{1}{n}$ then any KMS complex of type A over K is a γ'-local spectral expander, for $\gamma' = \frac{\gamma}{1-(n-1)\gamma}$.

For the cosystolic expansion result, we have to restrict to Dynkin diagrams on $(n+1)$ vertices which are n-classical. This means that any connected component of any subdiagram on n vertices is of type A_k, B_k, C_k, or D_k as depicted in Figure 1.4.

Main Theorem B (Corollary 6.5.4)**.** Let $n \geq 3$ be an integer and q a prime power. Let A be a Dynkin diagram on $n+1$ vertices which is n-classical. Let X be a KMS complex of type A over $\mathbb{F}_q$. Let Y denote the $(n-1)$-skeleton of X. Then there exist $\varepsilon, \mu > 0, q_0 \in \mathbb{N}$ such that if $q \geq q_0$ then Y is an $(\varepsilon, \mu, \mathbb{A})$-cosystolic expander for every finitely generated abelian group $\mathbb{A}$.

The $(n-1)$-skeleton of a simplicial complex is the complex we obtain by only considering simplices up to dimension $(n-1)$ and forgetting the higher dimensional structure.

We also get a bound on the cosystolic expansion in degree 1 for arbitrary, also non-abelian, groups.

Main Theorem C (Corollary 6.5.6)**.** Let $n \geq 3$ be an integer. There exist $\varepsilon > 0, q_0 \in \mathbb{N}$ such that for all prime powers $q > q_0$ and every n-dimensional, n-classical KMS complex X over $\mathbb{F}_q$ and every group Λ, we have $h^1_{cs}(X, \Lambda) \geq \varepsilon$.

For classical congruence KMS complexes, we can sharpen the result and obtain 1-coboundary expansion.

Main Theorem D (Corollary 6.5.7)**.** Let $n \geq 3$ and let T_n denote one of the types A_n, B_n, C_n or D_n. Then there exist $q_0 \in \mathbb{N}, \varepsilon > 0$ such that

(i) for every field K with $|K| \geq q_0$ and $\mathrm{char}(K) = p \neq 2$,

(ii) for every group Λ with no non-trivial elements of order p, (e.g. if Λ is finite and $|\Lambda| < p$),

(iii) for every irreducible polynomial $f \in K[t]$ of degree at least 2

the congruence KMS complex $X(T_n, K, f)$ is a (Λ, ε) 1-coboundary expander.

For all three types of high-dimensional expansion that we covered, local spectral, cosystolic and 1-coboundary, KMS complexes give rise to new examples of infinite families of bounded degree, since in the all three cases, the expansion and the degree of the KMS complexes does only depend on the type/the Dynkin diagram of the group, and the field, but not on the finite quotient. Thus, after fixing a type and a field satisfying the assumptions of the result that we want to apply, we can take an infinite family of finite quotients of $\mathcal{U}_A(K)$ which give rise to KMS complexes and then these will be families of HDX as desired.

1.4 Proof methods

To prove both local spectral and cosystolic expansion, we will use local-to-global results that were developed previously in [Opp18] (for local spectral) and [EK24] (for cosystolic expansion). In both cases we only have to consider the links of certain simplices and show good expansion for those.

Definition 1.4.1. Let X be a simplicial complex and let $\sigma \in X$. The link of σ in X is

$$\mathrm{lk}_X(\sigma) = \{\tau \in X \mid \sigma \cup \tau \in X, \sigma \cap \tau = \emptyset\}.$$

If X is pure n-dimensional then $\dim(\mathrm{lk}_X(\sigma)) = n - \dim \sigma - 1$.

We call a link *proper* if it has dimension at least 1 and it is not the whole complex, i.e. if $0 \leq \dim \sigma \leq n - 2$,

A key property of the KMS complexes is that the proper links do not depend on the finite quotient $\phi : \mathcal{U}_A(K) \to G$ but only on $\mathcal{U}_A(K)$. Furthermore, the links are isomorphic to certain subcomplexes of spherical buildings called opposite complexes. Spherical buildings are highly symmetric simplicial complexes. Intuitively, they can be thought of as a bouquet of triangulated spheres (and higher dimensional generalizations thereof) such that each two simplices are contained in a common sphere. For each type from Figure 1.4 we get one way of triangulating the sphere (or its higher dimensional analogs) and hence a corresponding building. In each triangulated sphere we have an opposition relation, which intuitively says which simplices lie on opposite sides of the sphere. Thus, after fixing one maximal simplex, we can consider the complex of all simplices opposite the fixed one. This is called the opposite complex.

Each Chevalley group over a finite field gives rise to a finite, spherical building. We demonstrate this again using SL_3 as example Chevalley group.

Example 1.4.2. Let K be a finite field. Consider the following subgroups inside $\mathrm{SL}_3(K)$:

$$P_1 = \begin{pmatrix} K & K & K \\ 0 & K & K \\ 0 & K & K \end{pmatrix}, \ P_2 = \begin{pmatrix} K & K & K \\ K & K & K \\ 0 & 0 & K \end{pmatrix}.$$

Then $\Delta = \mathcal{CC}(\mathrm{SL}_3(K); (P_1, P_2))$ is the spherical building of type A_2 over K. The complex opposite the edge $\{1P_1, 1P_2\}$ has the form

$$\Delta^{\mathrm{op}} = \mathcal{CC}\left(\begin{pmatrix} 1 & 0 & 0 \\ K & 1 & 0 \\ K & K & 1 \end{pmatrix}; \left(\begin{pmatrix} 1 & 0 & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & K & 1 \end{pmatrix} \right) \right).$$

We can also describe the building from an incidence geometric viewpoint. Let $V = K^3$ be the 3-dimensional vector space over K. Then the vertices of Δ correspond to proper subspaces $0 < U < V$ and two such distinct vertices U, W are connected by an edge if they are incident, i.e. if $U \leq W$ or $W \leq U$.

We can go from one to the other description as follows. Note that $\mathrm{SL}_3(K)$ acts transitively on the set of 1-dimensional and the set of 2-dimensional subspaces of K^3. We have that $P_1 = \mathrm{Stab}_{\mathrm{SL}_3(K)}(\langle (1,0,0)^T, (0,1,0)^T \rangle_K)$ and $P_2 = \mathrm{Stab}_{\mathrm{SL}_3(K)}(\langle (1,0,0)^T \rangle_K)$. Thus $\{0 < U < V\} \cong \mathrm{SL}_3(K)/P_1 \sqcup \mathrm{SL}_3(K)/P_2$. The action of $\mathrm{SL}_3(K)$ preserves incidence, and thus the adjacency relation in both graphs is the same.

To show local spectral expansion of the KMS complexes, Oppenheim's Trickling Down Theorem (Theorem 2.3.6) reduces the problem to showing that the complexes themselves are connected and that the links of codimension 2 faces are good expander graphs. The connectedness follows easily from the group theoretic properties. The expansion of the links, which in this case are graphs, can be deduced from the known expansion properties of the spherical buildings in dimension 1.

To prove cosystolic expansion the local-to-global result by Evra-Kaufman reduces the question to showing that the complex is a good local spectral expander, and that all proper links are coboundary expanders, with coboundary expansion constant independent of the spectral expansion constant. A large part of our work is devoted to showing coboundary expansion of the links. The arguably only way for showing coboundary expansion at this point is the cones method. In the graph case, a cone function is a family of paths from one fixed vertex v to all other vertices. The cone radius is then the maximum over the length of these paths. If the graph is symmetric enough, this indeed gives rise to a bound on the expansion, as the following theorem shows.

Theorem 1.4.3 ([Chu97, Theorem 7.1])**.** Let X be a finite connected graph such that its automorphism group acts transitively on the edges. Let D denote its diameter, i.e. $D = \max_{u,v\in X(0)} \min\{\text{length}(P) \mid P \text{ path between } u \text{ and } v\}$ and let $h(X)$ be its Cheeger constant. Then $h(X) \geq \frac{1}{2D}$.

Higher dimensional cone functions are a generalization of this idea such that for a sufficiently symmetric complex a bound on the cone radius gives a bound on the coboundary expansion constant.

We develop new tools for constructing cone functions which allow us to build cone functions for opposite complexes of spherical buildings. We follow the work of Abramenko in [Abr96], where he showed that the opposite complexes are highly connected, thus quantifying his result. This uses the incidence geometric viewpoint on buildings.

Lastly, we show that the first cohomology group of the classical congruence KMS complexes is trivial with respect to many coefficient groups, which, together with the cosystolic expansion results, implies that these KMS complexes are 1-coboundary expanders.

1.5 Outline

The thesis starts with two chapters on preliminaries, which cover the two main topics of this work, high-dimensional expansion and Kac–Moody theory.

In Chapter 2, we start by recalling some basic facts on expander graphs in Section 2.1. Then we discuss simplicial complexes and define their cohomology and homology both with abelian and non-abelian coefficient groups in Section 2.2. Using the language of simplicial complexes, we define various notions of high-dimensional expanders, namely spectral, local spectral, geometric and topological, coboundary and cosystolic in Section 2.3. In Section 2.4 we explain a main tool for constructing simplicial complexes from groups, the coset complexes. We finish the chapter by describing the known constructions of high-dimensional expanders previous to our work in Section 2.5.

Chapter 3 covers the preliminaries on Kac–Moody theory. We start by recalling some facts about Lie algebras and their associated root systems in Section 3.1, and describe the associated Chevalley and Steinberg groups in Section 3.2. We then generalize the ideas of semisimple Lie algebras to the infinite dimensional Kac–Moody algebras in Section 3.3. Using these tools we define the minimal Kac–Moody group associated to a Kac–Moody algebra in Section 3.4. A key tool for working with Kac–Moody groups is their action on the associated twin building. We describe these buildings, both from a simplicial complex viewpoint as well as the chamber system approach in Section 3.5. This allows us to carefully study the opposition relation both in spherical buildings and in twin buildings. The complex opposite a fixed chamber will play a crucial role later in the proofs, thus we describe it in detail and show how it interacts with the maximal unipotent subgroup U^+ of the Kac–Moody group in Section 3.6. We finish the chapter with a description of the buildings of type A_n, C_n, and D_n as flag complexes of certain incidence geometries in

Section 3.7. This will be important for the proof of coboundary expansion of the opposite complexes, but is not relevant for other parts of our work.

In Chapter 4, we introduce the protagonists of this work, the KMS groups in Section 4.1, and KMS complexes in Section 4.2. We describe a concrete example, the congruence KMS complexes and KMS complexes in matrix groups of arbitrary large rank in Section 4.3. We finish the chapter by showing that KMS complexes are local spectral expanders in Section 4.4. These results were mostly covered in [GV25].

In Chapter 5, we introduce cone functions, a key tool in showing coboundary expansion both with abelian and non-abelian coefficients. We give detailed proofs for some known results and develop several new tools for constructing cone functions which were developed in [OV25a; OV25b].

In Chapter 6, we put all the pieces together and show that classical KMS complexes give rise to cosystolic expanders. We achieve this by constructing cone functions for the opposite complexes of type A_n, C_n and D_n in Sections 6.1 to 6.3, respectively. In Section 6.4 we show vanishing of the first cohomology of the congruence KMS complexes. In Section 6.5, we use the local-to-global machinery to conclude cosystolic expansion and 1-coboundary expansion.

Lastly, in Chapter 7, we discuss further research questions and ideas around the topic of high-dimensional expansion and KMS complexes.

1.6 Conventions

We denote by $\mathbb{N} = \{1, 2, 3, \dots\}$ the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. As customary, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote the integers, rational, real and complex numbers respectively. For a prime power q, we denote the field with q elements by $\mathbb{F}_q$. For a ring R, we denote by $R^\times$ the set of (multiplicatively) invertible elements. For $a \in \mathbb{Z}$ we write $\mathbb{Z}_{\geq a} = \{z \in \mathbb{Z} \mid z \geq a\}$ and similarly for $\leq$ and $\mathbb{N}, \mathbb{R}$. For a set A we let $|A|$ be the cardinality of A. For a vector space V over a field K its dual is denoted as $V^* = \{f : V \to K \text{ linear}\}$.

For two indices i, j in some index set, we denote by δ_{ij} is the Kronecker delta, i.e.

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}.$$

We will use $A := B$ if we want to stress that A is defined as B, but if it is clear that we mean a definition, we also just write $A = B$.

Chapter 2

Preliminaries - High-dimensional expanders

We collect the necessary preliminaries on expander graphs, simplicial complexes and their (co)homology, high-dimensional expansion and coset complexes. We then discuss constructions of high-dimensional expanders previous to our work.

2.1 Expander graphs

Thinking of expander graphs one should have the picture of an optimal communication network in mind. On one hand, information in the network should spread quickly and cutting a few edges should not separate a significant part from the rest, on the other hand, it should be cost efficient, hence be sparse and not contain too many edges. Two main ideas for measuring the connectivity of a graph have developed. The Cheeger constant measures how many edges need to be cut to separate a large fraction of vertices. The spectral gap gives information on the convergence rate of the (lazy) random walk on the graph, hence measuring how fast information can spread. Luckily, we don't have to decide for one of the two methods to define expander graphs, since the Cheeger inequality relates the two notions. The sparseness of the graphs is enforced by not just considering one graph, but an infinite family of finite graphs with uniform bound on the degree, but growing number of vertices.

For this section, let $G = (V, E)$ denote a finite, simplicial graph (i.e. no orientation, loops or double edges). We equip the graph with a weight function $w : V \cup E \to \mathbb{R}^+$. For a subset $A \subseteq V \cup E$ we denote $\|A\| = \sum_{a \in A} w(a)$. Further, for two subsets $A, B \subseteq V$ we write $E(A, B) = \{\{v, u\} \in E \mid v \in A, u \in B\}$ for the set of edges between the sets A and B.

Definition 2.1.1. Let $G = (V, E)$ be a finite, simplicial graph with weight function w. Its *Cheeger constant* is defined as

$$h(G) = \min_{\emptyset \neq A \subsetneq V} \frac{\|E(A, V \setminus A)\|}{\min(\|A\|, \|V \setminus A\|)}.$$

The usual definition of the Cheeger constant is then obtained by assigning weight 1 to each vertex and edge.

Let M denote the random walk matrix of G, i.e. M is indexed by the vertices and its entries are given, for $v, u \in V$, by

$$M_{v,u} = \frac{w(\{v,u\})}{\sum_{e \in E : v \in e} w(e)}.$$

Here we set $w(\{v,u\}) = 0$ if $\{v,u\} \notin E$. $M_{v,u}$ denotes the transition probability of moving from v to u in one step of the random walk. This matrix is self-adjoint with respect to the inner product $\langle a, b \rangle = \sum_{v \in V} \left(\sum_{e \in E : v \in e} w(e) \right) a_v b_v$ for $a, b \in \mathbb{R}^{|V|}$. Hence all the eigenvalues of M are real and we denote them by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{|V|}$. Since M is a stochastic matrix, we have $\lambda_1 = 1$. The value $1 - \lambda_2$ is called the *spectral gap* of G. We will write $\lambda_2(G)$ to denote the second largest eigenvalue of the random walk matrix of G.

Proposition 2.1.2. (Cheeger inequality, see e.g. [KS11, Proposition 1.84].) For any finite, simplicial, d-regular graph $G = (V, E)$ with constant 1 weight function, we have

$$\frac{h(G)^2}{2} \leq \frac{1 - \lambda_2(G)}{d} \leq 2h(G).$$

Definition 2.1.3. Let $(G_n)_{n \in \mathbb{N}}$ be family of finite, simplicial graphs with respective vertex sets $V(G_n)$. We call it a *family of expanders* if there exist $d \in \mathbb{N}, \varepsilon > 0$ such that

(i) for all n, $h(G_n) \geq \varepsilon$, or equivalently $1 - \lambda_2(G_n) \geq \varepsilon$,

(ii) for all n, for all $v \in V(G_n)$ we have $\deg(v) \leq d$,

(iii) $|V(G_n)| \to \infty$ for $n \to \infty$.

More on the history, applications and constructions of expander graphs can be found e.g. in [HLW06].

2.2 Simplicial complexes and their (co)homology

We extend the setting of graphs to higher dimensions by allowing three vertices to form a triangle, four to form a tetrahedron etc. which gives rise to objects called simplicial complexes. To be able to later describe the generalizations of expansion to simplicial complexes, we need the language of cohomology which we also introduce in this chapter.

Simplicial complexes We start by giving the definition of simplicial complexes and recall some basic properties.

Definition 2.2.1. A *simplicial complex* X consists of a set of vertices V and a collection of subsets of vertices $X \subseteq \mathcal{P}(V)$ such that

(i) $\{v\} \in X$ for all $v \in V$,

(ii) for all $A \in X, B \subset A$ we have $B \in X$ (i.e. X is closed under taking subsets).

The elements of X are called simplices or faces. Given a simplex $\sigma \in X$ we define its dimension $\dim(\sigma) = |\sigma| - 1$. We will denote $X(k) = \{\sigma \in X \mid \dim(\sigma) = k\}$ for all $k \in \mathbb{N}_0$. We set $X(-1) = \{\emptyset\}$.

As is customary, we will identify the singletons $\{v\}$ with the vertices $v \in V$. In particular, we identify $X(0)$ with V and also write $X(0)$ for the set of vertices.

We call a simplicial complex finite if its vertex set is finite. We call it n-dimensional for some $n \in \mathbb{N}$ if $\max_{\sigma \in X} \dim(\sigma) = n$. Further, X is pure n-dimensional if every face is contained in an n-dimensional one.

The *degree* of a simplex σ in a pure n-dimensional complex X is the number of maximal faces containing σ, i.e. $\deg(\sigma) = |\{\tau \in X(n) \mid \sigma \subseteq \tau\}|$.

The *k-skeleton* of X, for some $1 \leq k \leq n$, is $X^{\leq k} = \bigcup_{i=0}^{k} X(i)$. This is a k-dimensional simplicial complex obtained by forgetting the higher-dimensional structure of X. In particular, the 1-skeleton of X is its underlying graph.

We call a simplicial complex *connected* if its 1-skeleton is connected as a graph.

Definition 2.2.2. Given a simplicial complex X and a simplex $\sigma \in X$, the *link* of σ in X is defined by

$$\mathrm{lk}_X(\sigma) = \{\tau \in X \mid \tau \cap \sigma = \emptyset, \tau \cup \sigma \in X\}.$$

Remark 2.2.3. Intuitively, the link describes how the complex locally behaves around the fixed simplex. If $\dim(\sigma) = k$ and X is pure n-dimensional then $\mathrm{lk}_X(\sigma)$ will be an $(n-k-1)$-dimensional simplicial complex. As we will see below, this is a useful tool to reduce the dimension and work with inductive arguments.

Similarly to the link, one can also consider the star around a simplex.

Definition 2.2.4. Let X be a simplicial complex, $\sigma \in X$. Then the *star* in X around σ is

$$\mathrm{st}_X(\sigma) = \{\tau \in X \mid \sigma \subseteq \tau\}.$$

Remark 2.2.5. The link and star in X around σ are related as follows.

$$\mathrm{lk}_X(\sigma) = \{\tau \setminus \sigma \mid \tau \in \mathrm{st}_X(\sigma)\}.$$

Remark 2.2.6. A different way to see a simplicial complex is as a partially ordered set (poset) $(A, \leq)$ with the property that for every two elements $a, b \in A$ there exists a greatest lower bound and such that for all $a \in A : \{b \in A \mid b \leq a\} \cong \mathcal{P}(V_a)$ for some set V_a. Such a poset always has one smallest element, corresponding to the empty set in the simplicial complex. The minimal elements of A without the smallest element correspond to the vertices. Conversely, a simplicial complex X with the inclusion relation is a poset satisfying the above properties.

This is not to be confused with the following construction that assigns to an arbitrary non-empty poset $(A, \leq)$ a simplicial complex $\mathrm{Flag}(A)$ with vertex set being the set A, and a where a subset $B \subset A$ forms a simplex if its elements can be arranged in a flag, i.e. of the form $b_1 \leq b_2 \leq \ldots$.

Simplicial (co)homology - abelian coefficients Homology and cohomology were developed as a tool to compute homotopy groups of topological spaces and has proven especially useful when applied to simplicial complexes due to their discrete nature. A standard reference for this topic is [Hat02].

Let $\mathbb{A}$ be an abelian group, X an n-dimensional simplicial complex. For $0 \leq i \leq n$, let $\overrightarrow{X}(i)$ denote the space of oriented i-dimensional simplices of X. Given a simplex $\{v_0, \dots, v_k\} \in X$ we will write $[v_0, \dots, v_k]$ for the oriented simplex given by the order of the vertices, i.e. the orbit of $(v_0, \dots, v_k)$ under the action of all even permutations on $k+1$ elements. Note that every simplex has two possible orientations, e.g. for a triangle one can read the vertices clockwise or anti-clockwise. Given $\sigma \in \overrightarrow{X}$ we write $-\sigma$ for the oriented simplex on the same vertex set but with the other orientation. Further note that for $[v_0, \dots, v_k] \in \overrightarrow{X}$ and a permutation $\phi \in \mathrm{Sym}(k+1)$ we have $[v_{\phi(0)}, \dots, v_{\phi(k)}] = (-1)^{\mathrm{sign}(\phi)}[v_0, \dots, v_k]$. Given oriented simplices $\tau = [v_0, \dots, v_k] \in \overrightarrow{X}(k)$ and $\sigma \in \overrightarrow{X}(k-1)$ we define their oriented incidence number $[\tau : \sigma]$ to be 0 if $\sigma \not\subseteq \tau$ and $(-1)^i$ if the vertex that is contained in τ but not in σ is v_i, i.e. if $\sigma = [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_k]$.

Definition 2.2.7. The set of *i-cochains* of X with coefficients in $\mathbb{A}$ is the set of anti-symmetric functions from $\overrightarrow{X}(i)$ to $\mathbb{A}$, i.e.

$$C^i(X, \mathbb{A}) = \{f : \overrightarrow{X}(i) \to \mathbb{A} \mid f(-\sigma) = -f(\sigma) \text{ for all } \sigma \in \overrightarrow{X}(i)\}.$$

The i-th coboundary map is given as follows.

$$d_i : C^i(X, \mathbb{A}) \to C^{i+1}(X, \mathbb{A}) : (d_i f)(\sigma) = \sum_{\tau \in \overrightarrow{X}(i)} [\sigma : \tau] f(\tau).$$

The concatenation $d_i \circ d_{i-1} = 0$ for all $0 \leq i \leq n-1$. Hence, we can define the following.

Definition 2.2.8. We define the set of *i-coboundaries* $B^i(X, \mathbb{A}) = \operatorname{Im} d_{i-1}$ and *i-cocycles* $Z^i(X, \mathbb{A}) = \ker d_i$. Then the i-th *cohomology group* is given by $H^i(X, \mathbb{A}) = Z^i(X, \mathbb{A})/B^i(X, \mathbb{A})$.

We will also need the dual versions of Definition 2.2.7 and Definition 2.2.8.

Definition 2.2.9. The set of *i-chains* of X with coefficients in $\mathbb{A}$ is the set of formal linear combinations of oriented i-simplices with coefficients in $\mathbb{A}$, i.e.

$$C_i(X, \mathbb{A}) = \left\{ \sum_{\sigma \in \overrightarrow{X}(i)} \lambda_\sigma \sigma \mid \lambda_{-\sigma} = -\lambda_\sigma \text{ for all } \sigma \in \overrightarrow{X}(i) \right\}.$$

The i-th *boundary map* is given as follows.

$$\partial_i : C_i(X, \mathbb{A}) \to C_{i-1}(X, \mathbb{A}) : \partial_i \left(\sum_{\sigma \in \overrightarrow{X}(i)} \lambda_\sigma \sigma \right) = \sum_{\tau \in \overrightarrow{X}(i-1)} \left(\sum_{\sigma \in \overrightarrow{X}(i)} [\sigma : \tau] \lambda_\sigma \right) \tau.$$

We again have that $\partial_i \circ \partial_{i+1} = 0$.

Definition 2.2.10. We define the set of *i-boundaries* $B_i(X, \mathbb{A}) = \operatorname{Im} \partial_{i+1}$ and *i-cycles* $Z_i(X, \mathbb{A}) = \ker \partial_i$. Then the i-th *homology group* is given by $H_i(X, \mathbb{A}) = Z_i(X, \mathbb{A})/B_i(X, \mathbb{A})$.

For finite simplicial complexes, the homology and cohomology group over a field are finite dimensional vector spaces which are dual to each other. Hence we observe the following.

Observation 2.2.11. Let X be a finite n-dimensional complex and $\mathbb{F}$ a field. Then $H^i(X, \mathbb{F}) \cong H_i(X, \mathbb{F})$ for all $0 \leq i \leq n$.

Simplicial cohomology - non-abelian version Here we recall the basic definitions of cohomology with potentially non-abelian coefficient group Λ. In this setting it only makes sense to consider $-1, 0$ and 1 cochains. We follow the exposition in [OV25b].

Throughout, we denote X to be a finite pure n-dimensional simplicial complex. We will write X_{ord} for the set of ordered simplices of X. This means that a simplex $\{v_0, \ldots, v_k\} \in X$ does not only get assigned an orientation as before, but we differentiate between all possible orderings of the vertices, i.e. all $(v_{\pi(0)}, \ldots, v_{\pi(k)})$ for all $\pi \in \operatorname{Sym}(k+1)$. Note that $\overrightarrow{X}(i) \cong X_{\text{ord}}(i)$ for $i \leq 1$. Like above, we will denote ordered simplices with round brackets, while we use square brackets for oriented simplices.

Given a group Λ, now potentially non-abelian, we define the space of k-cochains for $k = -1, 0, 1$ with values in Λ by

$$
\begin{aligned}
C^{-1}(X, \Lambda) &= \{\phi : \{\emptyset\} \to \Lambda\}, \\
C^0(X, \Lambda) &= \{\phi : X(0) \to \Lambda\}, \\
C^1(X, \Lambda) &= \{\phi : X_{\text{ord}}(1) \to \Lambda \mid \forall (u, v) \in X_{\text{ord}}(1), \phi((u, v)) = (\phi((v, u)))^{-1}\}.
\end{aligned}
$$

We define the coboundary maps $d_k : C^k(X, \Lambda) \to C^{k+1}(X, \Lambda)$ as follows:

- $k = -1$: For every $\phi \in C^{-1}(X, \Lambda)$ and every $v \in X(0)$, we define $d_{-1}\phi(v) = \phi(\emptyset)$.
- $k = 0$: For every $\phi \in C^0(X, \Lambda)$ and every $(v_0, v_1) \in X_{\text{ord}}(1)$, we define $d_0\phi((v_0, v_1)) = \phi(v_1)\phi(v_0)^{-1}$.
- $k = 1$: We also define the map $d_1 : C^1(X, \Lambda) \to \{\phi : X_{\text{ord}}(2) \to \Lambda\}$ as follows: for every $\phi \in C^1(X, \Lambda)$ and every $(v_0, v_1, v_2) \in X_{\text{ord}}(2)$, we define $d_1\phi((v_0, v_1, v_2)) = \phi((v_0, v_1))\phi((v_2, v_0))\phi((v_1, v_2))$.

Remark 2.2.12. Note that for $\phi \in C^1(X, \Lambda)$ we have that $d_1\phi$ is not anti-symmetric anymore, i.e. $d_1\phi((v_{\pi(0)}, v_{\pi(1)}, v_{\pi(2)})) \neq (d_1\phi((v_0, v_1, v_2)))^{\operatorname{sign}(\pi)}$ in general. Thus it is not a well-defined function on the oriented triangles as in the abelian case, and we have to consider ordered triangles instead.

But for $\pi \in \mathrm{Sym}(3)$ we still have that $d_1\phi((v_{\pi(0)}, v_{\pi(1)}, v_{\pi(2)})) = 1 \iff d_1\phi((v_0, v_1, v_2)) = 1$ since

$$
\begin{aligned}
d_1\phi((v_1, v_2, v_3)) = \phi((v_1, v_2))\phi((v_3, v_1))\phi((v_2, v_3)) = 1 &\qquad (2.1)\\
\iff \phi((v_3, v_1))\phi((v_2, v_3))\phi((v_1, v_2)) = 1 &\qquad (2.2)\\
\iff \phi((v_2, v_3))\phi((v_1, v_2))\phi((v_3, v_1)) = 1 &\qquad (2.3)\\
&\qquad (2.4)
\end{aligned}
$$

which covers all even permutations. For the odd ones, we can do the same trick applied to $(d_1\phi((v_1, v_2, v_3)))^{-1} = 1$. Thus we can still define $\mathrm{supp}(d_1\phi) = \{\{v_0, v_1, v_2\} \in X(2) \mid d_1\phi((v_0, v_1, v_2)) \neq 1\}$.

Note that we have

$$(d_0 \circ d_{-1})(\phi)((v_0, v_1)) = (d_{-1}\phi(v_1))(d_{-1}\phi(v_0))^{-1} = \phi(\emptyset)\phi(\emptyset)^{-1} = 1$$

and

$$
\begin{aligned}
(d_1 \circ d_0)(\phi)((v_0, v_1, v_2)) &= d_0\phi((v_0, v_1))d_0\phi((v_2, v_0))d_0\phi((v_1, v_2))\\
&= \phi(v_1)\phi(v_0)^{-1}\phi(v_0)\phi(v_2)^{-1}\phi(v_2)\phi(v_1)^{-1} = 1.
\end{aligned}
$$

Thus we can again define cocycles and coboundaries for $k = 0, 1$ by

$$
\begin{aligned}
Z^k(X, \Lambda) &= \{\phi \in C^k(X, \Lambda) : d_k\phi \equiv 1\},\\
B^k(X, \Lambda) &= \mathrm{Im}(d_{k-1})
\end{aligned}
$$

and get that $B^k(X, \Lambda) \subseteq Z^k(X, \Lambda)$ (as sets).

We further define an action of $C^{k-1}(X, \Lambda)$ on $Z^k(X, \Lambda)$:

- An element $\psi \in C^{-1}(X, \Lambda)$ acts on $Z^0(X, \Lambda)$ by
$$\psi.\phi(v) = \psi(\emptyset)^{-1}\phi(v), \forall \phi \in Z^0(X, \Lambda), \forall v \in X(0).$$
- An element $\psi \in C^0(X, \Lambda)$ acts on $Z^1(X, \Lambda)$ by
$$\psi.\phi((v_0, v_1)) = \psi(v_1)\phi((v_0, v_1))\psi(v_0)^{-1}, \forall \phi \in Z^1(X, \Lambda), \forall (v_0, v_1) \in X_{\mathrm{ord}}(1).$$

We denote by $[\phi]$ the orbit of $\phi \in Z^k(X, \Lambda)$ under the action of $C^{k-1}(X, \Lambda)$ and define the reduced k-th cohomology to be

$$H^k(X, \Lambda) = \{[\phi] : \phi \in Z^k(X, \Lambda)\}.$$

We will say that the k-th cohomology is trivial if $H^k(X, \Lambda) = \{[\phi \equiv 1]\}$ and note that the cohomology is trivial if and only if $B^k(X, \Lambda) = Z^k(X, \Lambda)$.

Remark 2.2.13. If Λ is an abelian group, the definitions in this section coincide with the ones in the previous section. Thus, this can be thought of as an extension of the abelian case. Note that in previous works on the topic $d_0\phi$ was defined to have the inverse value of how we define it, and $d_1\phi$ was adapted accordingly. Thus, it was not a direct generalization of the abelian case. For the purpose of considering coboundary and cosystolic expansion, it does not make a difference however, since in that case we consider the minima over all ordered simplices, thus for each value also its inverse will appear.

Topological viewpoint We briefly discuss how abstract simplicial complexes as defined above can be seen geometrically as subspaces of some $\mathbb{R}^d$. We follow [Mat03] and also refer to this source for more details and proofs.

Fix some dimension $d \in \mathbb{N}$ and $0 \leq n \leq d$. An n-simplex in $\mathbb{R}^d$ is a set

$$\sigma(v_0, \dots, v_n) = \left\{ \sum_{i=0}^{n} \lambda_i v_i \mid \sum_{i=0}^{n} \lambda_i = 1, \lambda_i \geq 0, i = 0, \dots, n \right\}$$

where $\{v_0, \dots, v_n\}$ is a set of points of $\mathbb{R}^d$ not contained in an $(n-1)$-affine subspace of $\mathbb{R}^d$. For any subset of points $\{v_{i_0}, \dots, v_{i_m}\} \subseteq \{v_0, \dots, v_n\}$, we call $\sigma(v_{i_0}, \dots, v_{i_m})$ a face or subsimplex of $\sigma(v_0, \dots, v_n)$. For a simplex $\sigma \in \mathbb{R}^d$ we write $\tilde{\sigma}$ for the underlying set of vertices, i.e. $\tilde{\sigma}(v_0, \dots, v_n) = \{v_0, \dots, v_n\}$.

A *geometric simplicial complex* consists of a set of vertices $V \subset \mathbb{R}^d$ and a set of simplices S inside $\mathbb{R}^d$ such that

(i) for each $\sigma, \tau \in S : \sigma \cap \tau$ is a subsimplex of both σ and τ,

(ii) when considering the underlying set $\tilde{S} = \{\tilde{\sigma} \mid \sigma \in S\}$ it is an abstract simplicial complex with vertex set V.

Definition 2.2.14. Let X be an abstract simplicial complex on the vertex set V. We call a geometric simplicial complex (V', S) in some $\mathbb{R}^d$ a geometric realization of X if $\tilde{S}$ is isomorphic to X as abstract simplicial complexes. We set $|X|_S = \bigcup_{s \in S} s \subseteq \mathbb{R}^d$ and view it as a topological space equipped with the subspace topology of $\mathbb{R}^d$.

Remark 2.2.15. Any finite simplicial complex X has a geometric realization in $\mathbb{R}^{|X(0)|}$. For each $v \in X(0)$ let e_v denote the corresponding standard basis vector in $\mathbb{R}^{|X(0)|}$. Then for $\sigma \in X$ we set

$$\sigma' = \left\{ \sum_{v \in \sigma \cap X(0)} \lambda_v e_v \mid \lambda_v \geq 0, \sum \lambda_v = 1 \right\}.$$

Then $(\{e_v \mid v \in X(0)\}, \{\sigma' \mid \sigma \in X\})$ is a geometric realization of X. Note that this is not optimal on the dimension of the vector space that we need. Indeed, any finite d-dimensional simplicial complex has a realization in $\mathbb{R}^{2d+1}$.

Remark 2.2.16. Any two geometric realizations of an abstract simplicial complex are homeomorphic. Thus it makes sense to talk about *the* geometric realization of an abstract simplicial complex X, and we write $|X|$ for one and hence up to homeomorphism any topological space associated to X as above.

Definition 2.2.17. Let $k \in \mathbb{Z}_{\geq 0}$. A topological space T is k-connected if for every $\ell = 0, \dots, k$, each continuous map from the ℓ-sphere to T, $f : S^\ell \to T$ is nullhomotopic, i.e. homotopic to a constant map that maps all points of S^ℓ to a single point $t_0 \in T$.

Remark 2.2.18. Let $k \geq 0$. A space T is k-connected if and only if all homotopy groups $\pi_i(T) = 0$ for $0 \leq i \leq k$ with respect to any basepoint.

Since homology and cohomology are homotopy invariant, we have that for a simplicial complex X such that $|X|$ is k-connected, then $H_i(X, \mathbb{A}) = 0$ and $H^i(X, \mathbb{A}) = 0$ for all $0 \leq i \leq k$. By the Hurewicz theorem we furthermore have $H_{k+1}(X, \mathbb{Z}) \cong \pi_{k+1}(|X|)$. We call an abstract simplicial complex k-connected if its geometric realization is k-connected.

Remark 2.2.19. A space T is called *contractible* if the identity map $\mathrm{id} : T \to T$ is nullhomotopic. This is equivalent to saying that T is homotopic to a one-point space. In this case, T is k-connected for all $k \in \mathbb{N}$.

2.3 Higher dimensional versions of expansion

The idea of high-dimensional expanders it to generalize the properties of being sparse and at the same time highly connected to simplicial complexes. There are various ways of how one can generalize being well connected to higher dimensions. We treat the main notions in the remainder of this section. In dimension ≥ 2, these notions are no longer related via a Cheeger inequality and hence have to be treated individually. Counterexamples can be found in [SKM14; GW16a].

To achieve sparseness, we ask, similar to the graph case (Definition 2.1.3), not just for one high-dimensional expander, but an infinite family of bounded degree complexes that grow in size but have a uniform bound on the degree in the following sense.

Definition 2.3.1. For any type of ε-high-dimensional expander (HDX) introduced below, we call a family of simplicial complexes $(X_m)_{m\in\mathbb{N}}$ a family of HDX (of that type) if there exist $n \in \mathbb{N}_{\geq 1}, d \in \mathbb{N}, \varepsilon > 0$ such that

- for each m, X_m is a finite, pure n-dimensional simplicial complex,
- for all m and all $v \in X_m(0)$ we have $\deg(v) \leq d$,
- $|X_m(0)| \to \infty$ for $m \to \infty$,
- for all m, X_m is an ε-HDX.

Spectral expansion We follow the description of spectral expansion in [Lub18]. For this part, we consider (co)homology with coefficients in $\mathbb{R}$. We fix a finite, pure, n-dimensional simplicial complex X. Now, $C^i(X, \mathbb{R})$ has a natural structure as a Hilbert space, given by the inner product

$$\langle f, g \rangle = \frac{1}{2} \sum_{\sigma \in \overrightarrow{X}(i)} \deg(\sigma) f(\sigma) g(\sigma)$$

for $f, g \in C^i(X, \mathbb{R})$. Here the degree of an oriented simplex is the same as the degree of the unoriented simplex on the same vertices. We can then consider the operators on $C^i(X, \mathbb{R})$ given by

$$\Delta_i^{up} = d_i^* d_i; \qquad \Delta_i^{down} = d_{i-1} d_{i-1}^*; \qquad \Delta_i = \Delta_i^{up} + \Delta_i^{down}$$

where d_i^* denotes the adjoint of d_i with respect to the inner product above. It has the following form, for $f \in C^{i+1}(X, \mathbb{R})$ and $\sigma \in \overrightarrow{X}(i)$:

$$(d_i^* f)(\sigma) = \frac{1}{\deg \sigma} \sum_{\tau \in \overrightarrow{X}(i+1)} [\tau : \sigma] \deg(\tau) f(\tau).$$

The operator Δ_i^{up} can be interpreted as describing a random walk in the following way. One walks on the i-dimensional faces of X. Standing at a face $\sigma \in X(i)$ one samples uniformly at random a face $\tau \in X(i+1)$ such that $\sigma \subseteq \tau$. Then, one samples a face $\sigma' \in X(i), \sigma' \subseteq \tau$ uniformly at random and walks from σ to σ'. The probability, given $\sigma, \sigma' \in X(i)$, to walk from σ to σ' in one step is then given by $\langle \mathbb{1}_\sigma, \Delta_i^{up} \mathbb{1}_{\sigma'} \rangle$, where $\mathbb{1}_\sigma : \overrightarrow{X}(i) \to \mathbb{R} \in C^i(X, \mathbb{R})$ has value 1 on σ, -1 on $-\sigma$ and 0 everywhere else. Similarly, Δ_i^{down} describes the transition probability for the down-walk, i.e. where the face τ is sampled among the $(i-1)$-dimensional faces.

Definition 2.3.2 ([Lub18, Definition 2.2])**.** The *i-dimensional spectral gap* of X is

$$\lambda^i(X) = \min\{\lambda \mid \lambda \text{ is an eigenvalue of } \Delta_i^{up}|_{(B^i(X,\mathbb{R}))^\perp}\}.$$

Remark 2.3.3. (i) Note that $B^i(X, \mathbb{R})^\perp = \ker d_{i-1}^*$. Thus we can rewrite $\lambda^i(X)$ as follows.

$$\lambda^i(X) = \inf_{f \in B^i(X,\mathbb{R})^\perp} \left\{ \frac{|\langle \Delta_i^{up} f, f \rangle|}{\langle f, f \rangle} \right\} = \left(\inf_{f \in \ker d_{i-1}^*} \left\{ \frac{\|d_i f\|}{\|f\|} \right\} \right)^2.$$

(ii) Note that Δ_i^{up} vanishes on $\ker(\Delta_i) \cong H^i(X, \mathbb{R})$ and thus $\lambda^i(X) > 0$ if and only if $H^i(X, \mathbb{R}) = 0$.

(iii) For a k-regular graph, $B^0(X, \mathbb{R})^\perp = \{f : X(0) \to \mathbb{R} \mid \langle f, \mathbb{1} \rangle = 0\}$ where $\mathbb{1}$ here denotes the constant 1 function. Thus, $\lambda^0(X)$ recovers the usual spectral gap in this case.

Definition 2.3.4. A pure n-dimensional simplicial complex X is called an *ε-spectral expander* if for every $0 \leq i \leq n-1$ we have $\lambda^i(X) \geq \varepsilon$.

The usual way of showing spectral expansion is by using a variant of Garland's method. It is described using the term of local spectral expansion.

For each $\tau \in X$ of dimension $k \leq n-2$, we denote the 1-skeleton of $\mathrm{lk}_X(\tau)$ by $X_\tau^{(1)}$. This graph comes naturally equipped with a weight function capturing the higher dimensional structure of X. It is given, for any edge $\{v, u\} \in X_\tau^{(1)}$, by

$$w_\tau(\{u, v\}) = (n-k-2)! \, |\{\sigma \in \mathrm{lk}_X(\tau)(n-1-k) \mid \{u, v\} \subseteq \sigma\}|.$$

Definition 2.3.5. Let X be a finite, pure n-dimensional simplicial complex, $0 \leq \varepsilon < 1$. We call X an *ε-local spectral expander* if for each $\tau \in X$ of dimension at most $n-2$, the second largest eigenvalue of the weighted random walk on the 1-skeleton of the link of τ is bounded by ε, i.e. $\lambda_2(X_\tau^{(1)}) \leq \varepsilon$.

Theorem 2.3.6 (Trickling Down Theorem [Opp18]). Let X be a pure n-dimensional, finite simplicial complex and $0 < \gamma < \frac{1}{n}$ such that

(a) $\mathrm{lk}_X(\tau)$ is connected for all $\tau \in X(i)$ for $i \leq n-2$;

(b) $\lambda_2(X_\tau^{(1)}) \leq \gamma$ for all $\tau \in X(d-2)$.

Then X is a $\left(\frac{\gamma}{1-(n-1)\gamma}\right)$-local spectral expander.

One can then apply Garland's method to deduce spectral expansion from local spectral expansion. More explicitly:

Theorem 2.3.7 (Garland, see [GW16a, Theorem 9]). Let X be a finite, pure n-dimensional simplicial complex such that for each $\tau \in X(n-2)$ we have that $\lambda_2(X_\tau^{(1)}) \leq 1-\varepsilon$, then

$$\lambda^{n-1}(X) \geq 1 + n\varepsilon - n.$$

Applying this theorem to the $(j+1)$-skeleton of X then gives the desired bound on $\lambda^j(X)$. In particular, local spectral expansion implies spectral expansion.

We also want to mention, that Garland's methods finds various applications in different areas as well, for example for proving property (T) (see e.g. [KK11]) or considering group stability (e.g. [De +20]).

Geometric and topological expansion We again follow [Lub18]. Let X be an n-dimensional pure simplicial complex. We call a map $f : X \to \mathbb{R}^n$ affine if every edge of X is mapped to the line segment between the images of its vertices, and similarly, a simplex σ is mapped to the convex hull of the images of its vertices. We say that f is continuous, when it is continuous considered as a map from the geometric realization of X to $\mathbb{R}^n$. For example, each edge is mapped to a continuous curve between the images of its vertices.

Definition 2.3.8. A pure n-dimensional simplicial complex X is an ε-*geometric* (resp. *topological*) expander if for every affine (resp. continuous) map $f : X \to \mathbb{R}^n$ there exists $z \in \mathbb{R}^n$ such that

$$\frac{|\{\sigma \in X(n) \mid z \in f(\sigma)\}|}{|X(n)|} \geq \varepsilon.$$

Remark 2.3.9.
(i) Gromov proved in [Gro10] that the complete simplex is a topological expander. He furthermore proved that spherical buildings over finite fields are topological expanders. But neither give rise to infinite families of bounded degree topological expanders.

(ii) The first infinite families of geometric expanders of bounded degree where constructed using the Ramanujan complexes and then extended to hold for further quotients of Bruhat-Tits buildings [Fox+12; Evr17].

(iii) Every expander graph gives rise to a 1-dimensional topological expander. But a graph does not need to be connected to be a topological expander, see the discussion in [Lub18].

Coboundary and cosystolic expansion The motivation for the definition of coboundary and cosystolic expansion was to find a method of proving topological expansion. But since then, it has shown to be interesting in its own right.

We again fix a finite, pure n-dimensional simplicial complex X and we consider its cohomology, this time with respect to an arbitrary abelian group $\mathbb{A}$ as coefficients. We define the following weight function $w : X \to [0,1]$, called Garland weights, by setting, for $\sigma \in X(i)$:

$$w(\sigma) = \frac{|\{\tau \in X(n) \mid \sigma \subseteq \tau\}|}{\binom{n+1}{i+1}|X(n)|}.$$

For a cochain $f \in C^i(X, \mathbb{A})$ we define its support to be the set $\operatorname{supp}(f) = \{\sigma \in X(i) \mid f(\sigma) \neq e_{\mathbb{A}}\}$ where $e_{\mathbb{A}}$ is the unit of $\mathbb{A}$. Note that this is well defined, since the orientation of σ does not play a role when deciding if $f(\sigma) = e_{\mathbb{A}}$.

The support together with the weight function induces a norm on the space of cochains of X by, for $f \in C^i(X, \mathbb{A})$, setting

$$\|f\| = \sum_{\sigma \in \operatorname{supp}(f)} w(\sigma).$$

Sometimes, we will also need to talk about the weight of an oriented simplex which we define, for $[v_0, \ldots, v_k] \in \overrightarrow{X}, k \geq 1$, as $w([v_0, \ldots, v_k]) = \frac{1}{2} w(\{v_0, \ldots, v_k\})$.

Remark 2.3.10. (i) One could work with a variety of different weight functions, see [Wil22] for a more general treatment. But this is the most prominent choice, which in particular works nicely, since it imposes a probability distribution on each of the $X(i)$.

(ii) In case $\mathbb{A} = \mathbb{R}$, the norm we consider here is different from the one for spectral expansion above, since here we only consider if the value of f on a simplex is non-zero or not, while previously we also took the value of f into account.

Definition 2.3.11. Let X be a finite, pure n-dimensional simplicial complex, $0 \leq k \leq n-1$. The k-coboundary and cosystolic expansion constant of X (with coefficients in $\mathbb{A}$) are defined to be

$$h^k_{\text{cb}}(X, \mathbb{A}) = \min_{\phi \in C^k(X,\mathbb{A}) \setminus B^k(X,\mathbb{A})} \frac{\|d_k \phi\|}{\min_{b \in B^k(X,\mathbb{A})}(\|\phi - b\|)},$$
$$h^k_{\text{cs}}(X, \mathbb{A}) = \min_{\phi \in C^k(X,\mathbb{A}) \setminus Z^k(X,\mathbb{A})} \frac{\|d_k \phi\|}{\min_{z \in Z^k(X,\mathbb{A})}(\|\phi - z\|)},$$

respectively.

Remark 2.3.12. We note that by definition $h^k_{\text{cs}}(X, \mathbb{A}) \geq h^k_{\text{cb}}(X, \mathbb{A})$ and that by definition $h^k_{\text{cs}}(X, \mathbb{A}) > 0$. We also note that $h^k_{\text{cs}}(X, \mathbb{A}) = h^k_{\text{cb}}(X, \mathbb{A})$ if and only if $H^k(X, \mathbb{A}) = 0$ and that if $H^k(X, \mathbb{A}) \neq 0$, then $h^k_{\text{cb}}(X, \mathbb{A}) = 0$.

Remark 2.3.13. The coboundary expansion constant $h^0_{\mathrm{cb}}(X, \mathbb{F}_2)$ with coefficients in $\mathbb{F}_2$ is precisely the Cheeger constant of the underlying weighted graph. We can identify subsets of vertices $A \subseteq X(0)$ with $f \in C^0(X, \mathbb{F}_2)$ via the support of f. The space of 0-coboundaries contains the constant zero and one functions, i.e. $B^0(X, \mathbb{F}_2) = \{[v \mapsto 0], [v \mapsto 1]\}$. Furthermore, $\mathrm{supp}(d_0 f) = E(\mathrm{supp}(f), X(0) \setminus \mathrm{supp}(f))$. Hence we can express the Cheeger constant as

$$h(X) = \min_{f \in C^0(X,\mathbb{F}_2) \setminus B^0(X,\mathbb{F}_2)} \frac{\|d_0 f\|}{\min_{b \in B^0(X,\mathbb{F}_2)} \|f + b\|}.$$

Definition 2.3.14. Let X be a finite, pure n-dimensional simplicial complex, let $\varepsilon > 0$ be a constant and $\mathbb{A}$ an abelian group. We say that X is an $(\varepsilon, \mathbb{A})$*-coboundary expander* if for every $0 \leq k \leq n-1$, $h^k_{\mathrm{cb}}(X, \mathbb{A}) \geq \varepsilon$.

Also, for constants $\varepsilon > 0, \mu > 0$, we will say that X *is an* $(\varepsilon, \mu, \mathbb{A})$*-cosystolic expander* if for every $0 \leq k \leq n-1$, $h^k_{\mathrm{cs}}(X, \mathbb{A}) \geq \varepsilon$ and for every $\phi \in Z^k(X, \mathbb{A}) \setminus B^k(X, \mathbb{A})$, $\|\phi\| \geq \mu$.

We say that X is an $(\varepsilon, \mathbb{A})$ m-coboundary expander for some $m \leq n-1$ if $h^k_{\mathrm{cb}}(X, \mathbb{A}) \geq \varepsilon$ for all $0 \leq k \leq m$. We similarly define m-cosystolic expansion.

Remark 2.3.15. Let X be an n-dimensional finite, pure simplicial complex. Intuitively, whether X is an i-coboundary/cosystolic expander, for some $i < k$ should only depend on the k-skeleton, since this holds for the corresponding (co)homology groups. But the expansion depends on the norm that we choose on the cochains, and in particular the weights that we assign to the simplices. In this work, we restricted to the Garland weights. They satisfy that all maximal simplices have uniform weight $\frac{1}{|X(n)|}$ and that the sum $\sum_{\sigma \in X(i)} w(\sigma) = 1$ for all $0 \leq i \leq n$. But the weight of σ depends on its degree and hence the structure of X up to the maximal dimension.

If we now consider the k-skeleton of X and want to investigate its coboundary expansion, we also have to consider the Garland weights in $X^{\leq k}$, i.e.

$$w_k(\sigma) = \frac{|\{\tau \in X(k) \mid \sigma \subseteq \tau\}|}{|X(k)| \binom{k+1}{i+1}}.$$

We denote $\deg_k(\sigma) = |\{\tau \in X(k) \mid \sigma \subseteq \tau\}|$. We now compare $w(\sigma)$ and $w_k(\sigma)$. Note that they are equal for all σ if and only if $w(\tau) = \frac{1}{|X(k)|}$ for all $\tau \in X(k)$.

Claim 1: $\deg_k(\sigma) d \geq \deg(\sigma)$ where $d = \max_{\tau \in X(k)} \deg(\tau)$.

Proof: First, note that for $\sigma \in X(i), i \leq k$

$$\{\eta \in X(n) \mid \sigma \subseteq \eta\} = \bigcup_{\tau \in X(k): \sigma \subseteq \tau} \{\eta \in X(n) \mid \tau \subseteq \eta\}.$$

Thus counting the sizes of the sets we get

$$|\{\eta \in X(n) \mid \sigma \subseteq \eta\}| \leq \sum_{\tau \in X(k): \sigma \subseteq \tau} \deg(\tau) \leq \deg_k(\sigma) d.$$

Claim 2: $\deg_k(\sigma) \leq \deg(\sigma) \binom{n-i}{k-i}$.

Proof: Given $\eta \in X(n), \sigma \in X(i)$ with $\sigma \subseteq \eta$ we have that

$$|\{\tau \in X(k) \mid \sigma \subseteq \tau \subseteq \eta\}| = \binom{n-i}{k-i}$$

since to get such a τ we have to choose $k+1$ vertices of the $n+1$ vertices of η, but we have to choose the $i+1$ vertices from σ, thus leaving us the choice of $(k+1)-(i+1)$ vertices out of $(n+1)-(i+1)$. On the other hand, we have

$$\{\tau \in X(k) \mid \sigma \subseteq \tau\} = \bigcup_{\eta \in X(n): \sigma \subseteq \eta} \{\tau \in X(k) \mid \sigma \subseteq \tau \subseteq \eta\}.$$

Similarly, we get

$$\begin{aligned} \deg_k(\sigma) &= |\{\tau \in X(k) \mid \sigma \subseteq \tau\}| \\ &\leq \sum_{\eta \in X(n): \sigma \subseteq \eta} |\{\tau \in X(k) \mid \sigma \subseteq \tau \subseteq \eta\}| \\ &= \deg(\sigma) \binom{n-i}{k-i}. \end{aligned}$$

From these two claims we conclude, for $\sigma \in X(i+1), \tau \in X(i)$,

$$\frac{1}{d}\binom{n-i-1}{k-i-1}\frac{w(\sigma)}{w(\tau)} \leq \frac{w_k(\sigma)}{w_k(\tau)} \leq \frac{w(\sigma)}{w(\tau)} d \binom{n-i}{k-i}.$$

Note $w(\tau), w_k(\tau) \neq 0$ because the complex is pure. Since $\|f\| = \sum_{\sigma \in \mathrm{supp}(f)} w(\sigma)$ we have

$$h^i_{\mathrm{cb}}(X, \Gamma)\frac{1}{d}\binom{n-i-1}{k-i-1} \leq h^i_{\mathrm{cb}}(X^{\leq k}, \Gamma) \leq h^i_{\mathrm{cb}}(X, \Gamma) d \binom{n-i}{k-i}.$$

Thus in particular, if (X_s) is a family of bounded degree coboundary expanders in dimension i, then so is $(X_s^{\leq k})$ for $i < k$. The same also holds for cosystolic expansion.

The following local-to-global result plays a key role in showing cosystolic expansion for a complex. A first version for $\mathbb{A} = \mathbb{F}_2$ was proven in [EK24] and then generalized to arbitrary abelian coefficient groups in [KM18; DD24a].

Theorem 2.3.16. Let $n \geq 3$ be an integer, $\varepsilon' > 0$ a constant and $\mathbb{A}$ be an abelian group. There are constants $0 < \lambda < 1, \varepsilon > 0, \mu > 0$ such that the following holds: For every finite, pure n-dimensional simplicial complex X if

- the simplicial complex X is a λ-local spectral expander,

and

- for every $0 \leq k \leq n-2$ and every $\tau \in X(k)$, it holds that $\mathrm{lk}_X(\tau)$ is an $(\varepsilon', \mathbb{A})$-coboundary expander, then the $(n-1)$-skeleton of X is an $(\varepsilon, \mu, \mathbb{A})$-cosystolic expander.

Remark 2.3.17. Gromov showed in [Gro10] that the complete complex is a coboundary expander with $\mathbb{F}_2$ coefficients. He then showed that every $(\varepsilon, \mathbb{F}_2)$-coboundary expander is an ε'-topological expander (with ε' depending on ε and the dimension of the complex). But up to this point, no family of bounded degree coboundary expanders in dimension at least 3 is known. But the question of finding bounded degree topological expanders was solved, using the result of [DKW18], which shows that every cosystolic expander over $\mathbb{F}_2$ is a topological expander, too.

Coboundary and cosystolic expansion with non-abelian coefficients In addition to the previous definitions, we can also consider coboundary and cosystolic expansion in degree 0 and 1 for non-abelian coefficients. We again fix a finite, pure n-dimensional simplicial complex X with the Garland weights w as defined above. Fix a potentially non-abelian group Λ. Recall that we define the support of cochains similar to the above definition. More precisely, for $\phi \in C^0(X, \Lambda)$, we define $\operatorname{supp}(\phi) \subseteq X(0)$ as

$$\operatorname{supp}(\phi) = \{v \in X(0) : \phi(v) \neq 1\}.$$

Also, for $\phi \in C^1(X, \Lambda)$, we define

$$\operatorname{supp}(\phi) = \{\{u, v\} \in X(1) : \phi((u, v)) \neq 1\},$$

and

$$\operatorname{supp}(d_1\phi) = \{\{v_1, v_2, v_3\} \in X(2) : (d_1\phi)((v_1, v_2, v_3)) \neq 1\}.$$

With this notation, we define the following norms: for $\phi \in C^k(X, \Lambda)$, $k = 0, 1$, we define

$$\|\phi\| = \sum_{\tau \in \operatorname{supp}(\phi)} w(\tau).$$

Also, for $\phi \in C^1(X, \Lambda)$, we define

$$\|d_1\phi\| = \sum_{\tau \in \operatorname{supp}(d_1\phi)} w(\tau).$$

Lastly, for two maps $\phi, \psi \in C^k(X, \Lambda), k = 0, 1$, we define $\operatorname{dist}(\phi, \psi)$ as follows: For $k = 0$, we define

$$\operatorname{dist}(\phi, \psi) = w\left(\{v \in X(0) \mid \phi(v)(\psi(v))^{-1} \neq 1\}\right).$$

For $k = 1$, we define

$$\operatorname{dist}(\phi, \psi) = w\left(\{\{u, v\} \in X(1) \mid \phi((u, v))(\psi((u, v)))^{-1} \neq 1\}\right).$$

Definition 2.3.18. Using the notations above, we will define coboundary and cosystolic expansion constants as follows: For $k = 0, 1$, define

$$h^k_{\mathrm{cb}}(X, \Lambda) = \min_{\phi \in C^k(X,\Lambda) \setminus B^k(X,\Lambda)} \frac{\|d_k \phi\|}{\min_{\psi \in B^k(X,\Lambda)} \mathrm{dist}(\phi, \psi)},$$

$$h^k_{\mathrm{cs}}(X, \Lambda) = \min_{\phi \in C^k(X,\Lambda) \setminus Z^k(X,\Lambda)} \frac{\|d_k \phi\|}{\min_{\psi \in Z^k(X,\Lambda)} \mathrm{dist}(\phi, \psi)}.$$

Remark 2.3.19. Observe the following (for $k = 0, 1$):

- For any X, $h^k_{\mathrm{cs}}(X, \Lambda) > 0$.
- If $H^k(X, \Lambda)$ is non-trivial, then $h^k_{\mathrm{cb}}(X, \Lambda) = 0$.
- If $H^k(X, \Lambda)$ is trivial, then $h^k_{\mathrm{cb}}(X, \Lambda) = h^k_{\mathrm{cs}}(X, \Lambda)$.
- If Λ is abelian, then the definitions in this chapter coincide with the corresponding ones in the previous section.

Definition 2.3.20. For a constant $\beta > 0$, we say that

- The complex X is a (Λ, β) 1-*coboundary expander*, if $h^0_{\mathrm{cb}}(X, \Lambda) \geq \beta$ and $h^1_{\mathrm{cb}}(X, \Lambda) \geq \beta$.
- The complex X is a (Λ, β) 1-*cosystolic expander*, if $h^0_{\mathrm{cs}}(X, \Lambda) \geq \beta$, $h^1_{\mathrm{cs}}(X, \Lambda) \geq \beta$ and for every $\phi \in Z^1(X, \Lambda) \setminus B^1(X, \Lambda)$ it holds that $\|\phi\| \geq \beta$.

Remark 2.3.21. A lower bound on h^0_{cb} is usually easy to verify: If the 1-skeleton of X is a connected finite graph, then for any non-trivial group Λ, $h^0_{\mathrm{cb}}(X, \Lambda)$ is exactly the Cheeger constant of the weighted 1-skeleton (where each edge $\{u, v\}$ is weighted by $w(\{u, v\})$). Moreover, the Cheeger constant of the weighted 1-skeleton can be bounded by the spectral gap of the weighted random walk on the 1-skeleton, which in concrete examples can be bounded using the Trickling Down Theorem 2.3.6.

In [DD24a], the following result was proven, generalizing Theorem 2.3.16 to the non-abelian setting, and improving parameters over the result of [KM21].

Theorem 2.3.22 ([DD24a, Theorem 8])**.** Let $0 \leq \lambda < 1, \beta > 0$ be constants and Λ be a group. For a finite pure n-dimensional simplicial complex X with $n \geq 3$, if

- for every vertex v, $h^1_{\mathrm{cb}}(\mathrm{lk}_X(v), \Lambda) \geq \beta$
- and X is a λ-local spectral expander,

then

$$h^1_{\mathrm{cs}}(X, \Lambda) \geq \frac{(1-\lambda)\beta}{24} - e\lambda,$$

where $e \approx 2.71$ is the Euler constant.

2.4 Coset complexes

In this section we investigate a method to construct highly symmetric simplicial complexes from a group and a set of subgroups. These are called coset complexes. Many examples of high-dimensional expanders rely on this construction. We also investigate some properties of the complexes that can be deduced from properties of the group and its subgroups. Coset complexes are in one-to-one correspondence with simplicial complexes admitting a type-preserving action that is transitive on the maximal faces.

Definition 2.4.1. Let X be a pure n-dimensional simplicial complex, $k \in \mathbb{N}_{\geq 2}$. We call X k-partite, if there exists a partition of the vertex set $X(0) = \sqcup_{i \in I} V_i$ with $|I| = k$ such that no two vertices of the same type are adjacent (i.e. connected by an edge). Such a partition induces a type function type : $X \to \mathcal{P}(I)$ on X, by setting for $v \in V_i$: $\text{type}(v) = \{i\}$ and for an arbitrary $\sigma \in X$: $\text{type}(\sigma) = \cup_{v \in \sigma} \text{type}(v)$.

Definition 2.4.2. Let X, Y be simplicial complexes. A *homomorphism* from X to Y is a map $\varphi : X(0) \to Y(0)$ such that for any set of vertices $\sigma \subset X(0)$ we have $\sigma \in X \Rightarrow \varphi(\sigma) \in Y$. We call φ an *isomorphism* if it is bijective and for any set of vertices $\sigma \subset X(0)$ we have $\sigma \in X \iff \varphi(\sigma) \in Y$. We denote by $\text{Aut}(X)$ the group of automorphisms of X.

An action of a group G on a simplicial complex X can be seen as a homomorphism φ from G to $\text{Aut}(X)$. The action is called faithful if the homomorphism is injective. We will use the following notation for $g \in G, \sigma \in X$: $g.\sigma = \varphi(g)(\sigma)$.

Let X be an n-dimensional, $n+1$-partite simplicial complex with indexing set $I = \{0, \dots, n\}$ and G a group acting on X type-preservingly (i.e $\text{type}(\sigma) = \text{type}(g.\sigma)$ for all $g \in G, \sigma \in X$). Further assume that the action is transitive on $X(n)$ (i.e. for all $\sigma, \tau \in X(n)$ there exists $g \in G$ such that $g.\sigma = \tau$). Fix one maximal simplex $\{v_0, \dots v_n\} \in X(n)$ and set $H_i = \text{Stab}_G(v_i)$ the vertex stabilizers in G. Since the action is type-preserving, we have $\text{Stab}_G(\{v_{i_1}, \dots, v_{i_k}\}) = \cap_{j=1}^k H_{i_j}$. We denote this set by $H_{\{i_1, \dots, i_k\}}$. Since the action is transitive on the maximal faces, the simplex $\{v_0, \dots, v_n\}$ is a fundamental domain of the action, and the set of simplices of type $T \subset \{0, \dots, n\}$ can be identified with G/H_T.

This idea can also be reversed in the following sense.

Definition 2.4.3. Let G be a group and $\mathcal{H} = (H_0, \dots, H_n)$ be a family of subgroups of G. Then the coset complex $\mathcal{CC}(G; \mathcal{H})$ is defined to be the simplicial complex with

- vertex set $\bigsqcup_{i=0}^n G/H_i$
- a set of vertices $\{g_1 H_{i_1}, \dots, g_k H_{i_k}\}$ forms a $(k-1)$-simplex in $\mathcal{CC}(G; \mathcal{H})$ if and only if $\bigcap_{j=1}^k g_j H_{i_j} \neq \emptyset$ and all i_j's are distinct.

Remark 2.4.4. Given a simplex $\{g_1 H_{i_1}, \dots, g_k H_{i_k}\} \in \mathcal{CC}(G; \mathcal{H})$ we define its type to be the set of indices $\{i_1, \dots, i_k\}$. The coset complex $\mathcal{CC}(G; \mathcal{H})$ is a pure n-dimensional, $(n+1)$-partite, simplicial complex. The group G acts by $g. \{g_1 H_{i_1}, \dots, g_k H_{i_k}\} = \{g g_1 H_{i_1}, \dots, g g_k H_{i_k}\}$. This action is type-preserving and transitive on the maximal faces. The vertex stabilizers of the vertices $1H_i$ are precisely the subgroups H_i.

Indeed, every n-dimensional, $n+1$-partite simplicial complex with type-preserving group action which is transitive on the maximal faces can be described as a coset complex.

Remark 2.4.5. The coset complex is an instance of a nerve of the covering $\{gH_i \mid 0 \leq i \leq n, g \in G\}$ of $\bigsqcup_{i=0}^n G/H_i$, as in [AH93].

There is a strong connection between the group theoretic properties of the subgroups H_i and the connectivity properties of the coset complex.

Lemma 2.4.6 (see e.g. [HS19, Lemma 3.3])**.** Let G be a group with subgroups $\mathcal{H} = (H_0, \ldots, H_n)$. Then $\mathcal{CC}(G; \mathcal{H})$ is connected (i.e. the underlying graph is connected) if and only if $G = \langle H_0, \ldots, H_n \rangle$.

Proposition 2.4.7 ([AH93, Theorem 2.4])**.** Let G be a group with subgroups $\mathcal{H} = (H_0, \ldots, H_n)$. Then $\mathcal{CC}(G; \mathcal{H})$ is 1-connected if and only if G is the free product of the subgroups H_i amalgamated along their intersections, as defined in Definition 3.5.17.

As they will be useful for the analysis of the KMS complexes below, we will mention two further properties of coset complexes.

Lemma 2.4.8 (see e.g. [HS19, Lemma 3.4])**.** Let G be a group with subgroups $\mathcal{H} = (H_0, \ldots, H_n)$. Set $X = \mathcal{CC}(G; \mathcal{H})$ and let $\sigma \in X$ be a simplex of type $T \subset \{0, \ldots, n\}$. Then the link of σ in X can again be described as a coset complex:

$$\mathrm{lk}_X(\sigma) \cong \mathcal{CC}(H_T; (H_{T \cup \{i\}})_{i \in \{0, \ldots, n\} \setminus T}).$$

Recall that $H_T = \cap_{j \in T} H_j$.

Lemma 2.4.9. Let G be a group with subgroups $\mathcal{H} = (H_0, \ldots, H_n)$. Let $v = gH_i$ be a vertex in $X = \mathcal{CC}(G; \mathcal{H})$. Then $\deg(v) = |H_i / (\cap_{j=0}^n H_j)|$.

Proof. The maximal simplices correspond to elements of $G/(\cap_{j=0}^n H_j)$. Set $I = \{0, \ldots, n\}$ and $H_I = \cap_{j=0}^n H_j$. A maximal simplex hH_I contains the vertex $v = gH_i$ if and only if $g^{-1}h \in H_i$. Hence there are $|H_i / (\cap_{j=0}^n H_j)|$ many distinct such simplices. $\square$

Coset complexes also appear under different disguises in the literature. For example the basic construction in [BH99] in many situations outputs complexes with geometric realization homeomorphic to a coset complex. The idea also appears in the area of incidence geometry, under the name of coset geometry, see e.g. [Bue79].

2.5 Examples prior to our work

In this section, we present the main examples of high-dimensional expanders prior to our work. We follow the historical development.

Ramanujan complexes - quotients of buildings We start this section by discussing the expansion properties of spherical buildings. Spherical buildings are very symmetric and well behaved simplicial complexes, in many cases coming from a Chevalley group, see Section 3.5 for a definition and more details. If the size of the underlying field (the thickness of the building) is finite, then the spherical building is a finite simplicial complex, which is the case we assume here.

First, note that the links inside a spherical building are again spherical buildings of smaller subtypes. This makes buildings very convenient with regard to the local-to-global results in Theorems 2.3.6, 2.3.16 and 2.3.22. The spectra of buildings of dimension 1, i.e. when the buildings are graphs, are known.

The eigenvalues of the Laplacian matrix $\Delta = \mathrm{id} - M$ (where M is the random walk matrix) of the spherical buildings of rank 2 are computed in [Gar73, Proposition 7.10]. Hence we can deduce from [Gar73, Proposition 7.10] the following values for the spectrum of the random walk matrix depending on the type. Here, q is an arbitrary prime power.

Proposition 2.5.1. Let M be the random walk matrix of the spherical building of type A_2 over $\mathbb{F}_q$. Then the spectrum is

$$\operatorname{spec}(M) = \left\{1, \frac{\sqrt{q}}{q+1}, -\frac{\sqrt{q}}{q+1}, -1\right\}.$$

Proposition 2.5.2. Let M be the random walk matrix of the spherical building of type B_2 over $\mathbb{F}_q$. Then the spectrum is

$$\operatorname{spec}(M) = \left\{1, \frac{\sqrt{2q}}{q+1}, 0, -\frac{\sqrt{2q}}{q+1}, -1\right\}.$$

Proposition 2.5.3. Let M be the random walk matrix of the spherical building of type G_2 over $\mathbb{F}_q$. Then the spectrum is

$$\operatorname{spec}(M) = \left\{1, \frac{\sqrt{3q}}{q+1}, \frac{\sqrt{q}}{q+1}, 0, -\frac{\sqrt{q}}{q+1}, -\frac{\sqrt{3q}}{q+1}, -1\right\}.$$

Corollary 2.5.4. For every $0 < \lambda < 1$ there exists $q' \in \mathbb{N}$ such that for all primes powers $q \geq q'$ any spherical building over the field $\mathbb{F}_q$ is a λ-local spectral expander.

Proof. The one dimensional links inside a spherical building are either a complete bipartite graph or one of the three graphs from Propositions 2.5.1 to 2.5.3. Thus, Theorem 2.3.6 gives the desired result. □

Furthermore, it was shown that spherical buildings are coboundary expanders by Gromov in [Gro10], see also [LMM16]. We give the precise statement from [KM18] which generalizes the previous results to arbitrary abelian coefficient groups.

Theorem 2.5.5. [KM18, Theorem 4.1] Let Δ be a spherical building coming from a finite group G with BN-pair of rank $n+1$ (i.e. of dimension n). Let $\mathbb{A}$ be an abelian group. Then for all $0 \leq k \leq n-1$ we have

$$h^k_{\mathrm{cb}}(\Delta, \mathbb{A}) \geq (2^n \omega_n)^{-1}$$

where ω_n is the maximal size of a Weyl group of rank $n+1$.

It is important to note that spherical buildings do not form an infinite family of bounded degree complexes. Thus, they do not themselves solve the question for high-dimensional expansion. But they are good building blocks for further construction.

This idea was used in the construction of the Ramanujan complexes in [LSV05] and simultaneously in [Li04]. Roughly, the construction starts with one infinite Bruhat-Tits building $\mathcal{B}$ of type $\tilde{A}_n$ coming from $G = \mathrm{PGL}_n(\mathbb{F}_p((t)))$. Each cocompact lattice in $\Gamma \leq G$ then gives rise to a finite complex $\Gamma \backslash \mathcal{B}$ for which each link is a spherical building. An infinite family of such lattices then gives rise to an infinite family of bounded degree complexes. In particular, their (co-dimension 1)-skeletons were the first examples of a family of cosystolic and hence also topological expanders, see [EK24, Theorem 1.3]. On the other hand, it was shown in [KKL14] that the Ramanujan complexes do not give rise to coboundary expanders.

This was overcome in dimension 2 by considering similar quotients of Bruhat-Tits buildings of type $\tilde{C}_n$. The difference here is that lattices in Lie groups of type C_n satisfy the congruence subgroup property, which is crucial for showing the vanishing of $H^1(\Gamma \backslash \mathcal{B}, \Lambda)$ for certain finite groups Λ. In particular, it was shown in [CL25b, Theorem 1.9] that certain quotients of $\tilde{C}_n$ buildings give rise to a family of 2-dimensional bounded degree coboundary expanders with $\mathbb{F}_2$ coefficients. For dimension ≥ 3 this question remains open.

High-dimensional expansion of coset complexes The first use of the coset complex construction related to high-dimensional expansion was in [KO23]. There, Kaufman and Oppenheim showed that coset complexes over certain subgroups over the elementary matrices give rise to local spectral expanders.

Let R be an algebra, $n \in \mathbb{N}_{\geq 2}$. For $1 \leq i \neq j \leq n, r \in R$, we write $e_{i,j}(r)$ for the $n \times n$ matrix with ones on the diagonal, r at position i, j and zeros everywhere else. The group of elementary matrices $\mathrm{EL}_n(R)$ is then the group of matrices generated by $\{e_{i,j}(r) \mid 1 \leq i \neq j \leq n, r \in R\}$.

Now, fix a prime power q and let $\mathbb{F}_q[t]$ denote the polynomial ring over $\mathbb{F}_q$. For each $s > n$ set $G^{(s)} = \mathrm{EL}_{n+1}(\mathbb{F}_q[t]/(t^s))$ and set $G = \mathrm{EL}_{n+1}(\mathbb{F}_q[t])$. For each $1 \leq i \leq n+1$ we define

$$K_i = \langle e_{j,j+1}(a + bt) \mid j \in \{1, \ldots, n+1\} \setminus \{i\}, a, b \in \mathbb{F}_q \rangle$$

where the $j+1$ is taken modulo $n+1$, i.e. $e_{n+1,n+2}(r) = e_{n+1,1}(r)$. We denote by $K_i^{(s)}$ the image of K_i inside $G^{(s)}$ given by considering each entry modulo t^s. Note that $K_i \cong K_i^{(s)}$. Lastly, we set $X^{(s)} = \mathcal{CC}\left(G^{(s)}; (K_i^{(s)})_{i \in \{1,\ldots,n+1\}}\right)$.

Theorem 2.5.6. [KO23, Theorem 1.2] $X^{(s)}$ is a $\frac{1}{\sqrt{q}-(n-1)}$-local spectral expander. In particular, given $\lambda > 0$, we can take q large enough such that $\{X^{(s)} \mid s > n\}$ is a family of n-dimensional λ-local spectral expanders with uniformly bounded degree.

This result was later generalized in [KO21] to also show coboundary expansion of certain links and thus cosystolic expansion in dimension 2. There the complexes $X^{(s)}$ were considered for $n = 4$. Then for any vertex $v \in X^{(s)}(0)$ its link is isomorphic to a complex $X_{link,q}$ (using Lemma 2.4.8).

Theorem 2.5.7. [KO21, Theorem 1.21 and 1.22] For every odd prime power q, the complex $X_{link,q}$ is a $(\lambda, \mathbb{F}_2)$-coboundary expander, with λ a constant independent of q.

Thus it follows from Theorem 2.3.16 that for sufficiently large q, the 2-skeleton of $X^{(s)}$, denoted $Y^{(s)}$ is a cosystolic expander, and the family $\{Y^{(s)} \mid s > 4\}$ is a family of bounded degree cosystolic expanders.

In [KOW24] it was shown that a slight variation of the Kaufman-Oppenheim complexes give rise to 1-coboundary expanders over many coefficient groups. They consider the subgroups

$$\tilde{K}_i = \langle e_{j,j+1}(g) \mid j \in \{1, \dots, n+1\} \setminus \{i\}, g \in \mathbb{F}_q[t], \deg(g) \leq 3 \rangle$$

where the $j+1$ is taken modulo $n+1$, i.e. $e_{n+1,n+2}(r) = e_{n+1,1}(r)$. We again denote by $\tilde{K}_i^{(s)}$ the image of $\tilde{K}_i$ inside $G^{(s)} = \mathrm{EL}_{n+1}(\mathbb{F}_q[t]/(t^s))$ and $\tilde{X}^{(s)} = \mathcal{CC}(G^{(s)}; (\tilde{K}_i^{(s)})_{i \in \{1,\dots,n+1\}})$.

Theorem 2.5.8 (Theorem 6.3)**.** [KOW24] Let $n \in \mathbb{N}_{\geq 3}$ and Λ a finite group. There is a constant p' such that for every prime $p \geq p'$, the family $\{\tilde{X}^{(s)} \mid s > 3n\}$ of n-dimensional complexes over $\mathbb{F}_p$ is a family of bounded degree 1-coboundary expanders over Λ.

The construction of Kaufman and Oppenheim was generalized by O'Donnell and Pratt to not just work for elementary matrices, but for Chevalley groups of any type except G_2 in [OP22].

To describe their construction, we will use the terminology of root systems and Chevalley groups, which will be defined in detail later in Chapter 3, in particular Section 3.2.

Let A be an irreducible Cartan matrix, not of type G_2, with corresponding root system Φ and simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$. We define the following set of roots $\mathcal{S} = \Pi \cup \{-(\alpha_1 + \dots + \alpha_n)\}$. Note that if A is of type A_n then $\alpha_1 + \dots + \alpha_n$ is the highest root in Φ.

Fix a finite field $\mathbb{F}$ of odd characteristic p. We define for each $\alpha \in \mathcal{S}$ the subgroup

$$H_\alpha = \langle x_\beta(a + bt) \mid \beta \in \mathcal{S} \setminus \alpha, a, b \in \mathbb{F} \rangle \leq \mathrm{Chev}_A(\mathbb{F}[t]).$$

For each $m \in \mathbb{N}_{\geq 2}$ we fix an irreducible polynomial $f_m \in \mathbb{F}[t]$ of degree m. Then we can consider the finite group $G_m = \mathrm{Chev}_A(\mathbb{F}[t]/(f_m))$ as a quotient of $\mathrm{Chev}(\mathbb{F}[t])$. We furthermore denote by H_α^m the image of H_α in G_m. Note that similar to the Kaufman-Oppenheim construction, $H_\alpha \cong H_\alpha^m$. Setting $K_m = \mathcal{CC}(G_m; (H_\alpha^m)_{\alpha \in \mathcal{S}})$ we get the following results.

Theorem 2.5.9. [OP22, Theorem 3.6] For every $\sigma \in K_m(n-2)$ we have that $\lambda_2(K_m) \leq \sqrt{\frac{2}{p}}$.

In particular, the sequence $(K_m)_{m=1}^{\infty}$ forms a family of bounded degree local spectral expanders.

These results were extended in [OS24], where the authors showed that the above construction with respect to the Cartan matrix of type B_3 yields complexes who's 2-dimensional skeletons form a family of cosystolic expanders.

Another, similar, construction of local spectral expanders appeared in [DLZ23]. There, the authors considered the subgroups

$$K_1 = \left\{ \begin{pmatrix} 1 & at & ct^2 \\ 0 & 1 & bt \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_q \right\} \leq \mathrm{SL}_3(\mathbb{F}_q[t]),$$
$$K_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ ct^2 & 1 & at \\ bt & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_q \right\} \leq \mathrm{SL}_3(\mathbb{F}_q[t]),$$
$$K_3 = \left\{ \begin{pmatrix} 1 & at & 0 \\ 0 & 1 & 0 \\ bt & ct^2 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{F}_q \right\} \leq \mathrm{SL}_3(\mathbb{F}_q[t]).$$

They then fix an infinite family of irreducible polynomials $f_n \in \mathbb{F}_q[t]$ such that each polynomial is primitive, $\deg(f_n) \to \infty$ and $3 \nmid q^{\deg(f_n)} - 1$ for all n. They further set $G_n = \mathrm{SL}_3(\mathbb{F}_q[t]/(f_n))$ and K_i^n the image if K_i inside G_n. Then the family $X_n = \mathcal{CC}(G_n; (K_1^n, K_2^n, K_3^n))$ is a family of local spectral expanders.

We compare these constructions to the KMS complexes in Section 4.5.

Chapter 3

Preliminaries - Kac–Moody theory

Lie theory studies groups of symmetries of certain geometric and algebraic objects, their Lie algebras, the infinitesimal transformations, and the correspondence between them. Finite dimensional, simple, complex Lie algebras have been classified by Killing and Cartan. This yields a small list of finite dimensional Lie algebras indexed by the so called Cartan matrices. Generalizing the ideas of Lie theory to infinite dimensional Lie algebras has proven to be challenging but fruitful. Both analytic generalizations, by modelling Lie groups on infinite-dimensional spaces, and more algebraic constructions have been investigated. The latter lead to Kac–Moody theory, which we will introduce in this chapter.

We recall some facts on Lie algebras and their root systems, and discuss groups of Lie type, the discrete version of Lie groups, before introducing Kac–Moody algebras and their associated minimal Kac–Moody groups. We will then investigate geometric objects, called buildings, on which groups of Lie type and Kac–Moody groups act. In the following chapters, we will be particularly interested in the maximal unipotent subgroups of the Kac–Moody groups, thus we will have a detailed look at them and the corresponding part in the building, the opposite complex. In the last section, we describe spherical buildings of type A_n, C_n, D_n geometrically as flag complexes.

3.1 Lie algebras and root systems

We start by recalling some facts from classical Lie theory. We mostly follow [Mar18].

Definition 3.1.1. A *Lie algebra* is a vector space $\mathfrak{g}$ over a field K together with a map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

(i) $[\cdot,\cdot]$ is bilinear,

(ii) $[x, x] = 0$ for all $x \in \mathfrak{g}$,

(iii) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in \mathfrak{g}$ (Jacobi identity).

A subspace I of $\mathfrak{g}$ is called an *ideal* of $\mathfrak{g}$ if $[I, \mathfrak{g}] := \langle [i, g] \mid i \in I, g \in \mathfrak{g} \rangle_K \subseteq I$. We call a Lie algebra *simple* if it has no ideals other than the full and zero space.

For each element $x \in \mathfrak{g}$ we define a map $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ by setting $\operatorname{ad} x(y) = [x, y]$ for $y \in \mathfrak{g}$. A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Cartan subalgebra* If

(i) it is a subalgebra, i.e. $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$,

(ii) it is nilpotent, i.e. there exists $n \in \mathbb{N}$ such that $\underbrace{[[[\mathfrak{h}, \mathfrak{h}], \mathfrak{h}], \dots]}_{n \text{ times}} = 0$,

(iii) if $[x, h] \in \mathfrak{h}$ for all $h \in \mathfrak{h}$ then $x \in \mathfrak{h}$.

Any complex Lie algebra has a Cartan subalgebra and any two Cartan subalgebras are isomorphic.

Example 3.1.2. The set of traceless complex matrices $\mathfrak{g} = \mathrm{sl}_{n+1}(\mathbb{C}) = \{A \in \mathrm{Mat}_{n+1}(\mathbb{C}) \mid \mathrm{Tr}(A) = 0\}$ together with the Lie bracket $[A, B] = AB - BA$ is a simple complex Lie algebra. A Cartan subalgebra of $\mathfrak{g}$ consists of the traceless diagonal matrices. Note that defining $[A, B] = AB - BA$ is a general tool to define a Lie bracket on an associative algebra.

Definition 3.1.3. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$. For each α in the dual $\mathfrak{h}^*$ of $\mathfrak{h}$ we set $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \forall f \in \mathfrak{h} : [f, x] = \alpha(f)x\}$. The *root system of* $\mathfrak{g}$ is defined as the set

$$\Phi = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}.$$

Fact 3.1.4. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}$ has a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

We collect some classical results from Lie theory. We keep the notation from above.

(i) $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$.

(ii) For any $\alpha \in \Phi, x_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$ there exists an $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that the assignment

$$x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad x_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad [x_{-\alpha}, x_\alpha] \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

defines an isomorphism $\mathfrak{g}_\alpha \oplus \mathbb{C}\alpha^\vee \oplus \mathfrak{g}_{-\alpha} \to \mathrm{sl}_2(\mathbb{C})$ of Lie algebras. We set $\alpha^\vee = [x_{-\alpha}, x_\alpha]$ and call it the *coroot* of α.

(iii) For all $\alpha, \beta \in \Phi$ we have $\alpha(\beta^\vee) \in \mathbb{Z}$.

(iv) The set Φ admits a root basis $\Pi = \{\alpha_1, \dots, \alpha_\ell\} \subseteq \Phi$ such that for every $\alpha \in \Phi$ there exist unique $n_1, \dots, n_\ell \in \mathbb{N}$ and $\epsilon_\alpha \in \{-1, 1\}$ such that

$$\alpha = \epsilon_\alpha \sum_{i=1}^{\ell} n_i \alpha_i.$$

We call the $\alpha_i \in \Pi$ *simple roots*. The number of simple roots ℓ is called the *rank* of Φ. The integer

$$\mathrm{ht}(\alpha) = \epsilon_\alpha \sum_{i=1}^{\ell} n_i$$

is called the *height* of α with respect to Π.

Fixing a root basis gives a partition of Φ into positive and negative roots $\Phi^{\pm} = \{\alpha \mid \epsilon_\alpha = \pm 1\}$.

(v) For each $\alpha \in \Phi$ we consider the reflection

$$s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^* : \beta \mapsto \beta - \beta(\alpha^\vee)\alpha.$$

The group generated by all reflections $W = \langle s_\alpha \mid \alpha \in \Phi\rangle \leq \mathrm{GL}(h^*)$ is called the *Weyl group* of $\mathfrak{g}$. It is generated by the reflections given by the simple roots $s_i = s_{\alpha_i}$ and $(W, S = \{s_1, \dots, s_\ell\})$ is a Coxeter system, which we define in Section 3.5.

The Lie algebra $\mathfrak{g}$ is uniquely determined by its *Cartan matrix*

$$A = (\alpha_j(\alpha_i^\vee))_{i,j=1}^{\ell}.$$

Note that in particular all entries are integers, $\alpha_i(\alpha_i^\vee) = 2$ for all i and $\alpha_j(\alpha_i^\vee) \leq 0$ for all $i \neq j$. We make the connection between a Lie algebra and its Cartan matrix more precise with the following definition.

Definition 3.1.5. Given a Cartan matrix $A = (a_{ij})_{i,j=1}^{\ell}$, we define the complex Lie algebra $\mathfrak{g}_A$ by its presentation: There are 3ℓ generators $e_i, f_i, \alpha_i^\vee$ for all $1 \leq i \leq \ell$ and defining relations, for all $1 \leq i, j \leq \ell$

$$[\alpha_i^\vee, \alpha_j^\vee] = 0, \tag{3.1}$$
$$[\alpha_i^\vee, e_j] = a_{ij} e_j,\ [\alpha_i^\vee, f_j] = -a_{ij} f_j, \tag{3.2}$$
$$[f_i, e_j] = \delta_{ij}\alpha_i^\vee \tag{3.3}$$
$$(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0,\ (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0 \text{ for } i \neq j. \tag{3.4}$$

For example $\mathrm{ad}(e_i)^2(e_j) = [e_i, [e_i, e_j]]$.

Fact 3.1.6. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ with Cartan matrix A. Then $\mathfrak{g} \cong \mathfrak{g}_A$ as Lie algebras.

Example 3.1.7. The Cartan matrix of $\mathrm{sl}_4(\mathbb{C})$ is $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ and looks similar for higher dimensional $\mathrm{sl}_n(\mathbb{C})$. Recall that the Cartan subalgebra $\mathfrak{h}$ is the set of traceless diagonal matrices. Let E_{ij} denote the matrix in sl_n with entry 1 in position (i,j) and 0 else. The space of roots with respect to the Cartan subalgebra of diagonal matrices is given by

$$\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$$

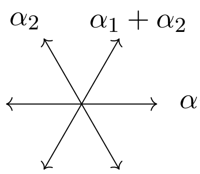


Figure 3.1: Visualization of the root system of type A_2.

where $\varepsilon_i \in \mathfrak{h}^*$ is given by requiring $\varepsilon_i(E_{jj} - E_{kk}) = \delta_{ij} - \delta_{ik}$. The set $\Pi = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1\}$ is a set of simple roots. The root system of $\mathrm{sl}_3(\mathbb{C})$ can be visualized in $\mathbb{R}^2$, see Figure 3.1. We can see the isomorphism from $\mathfrak{g}_A$ to $\mathrm{sl}_n(\mathbb{C})$ by mapping, for $1 \leq i \leq n-1$,

$$\begin{aligned} e_i &\to E_{i,i+1} \\ f_i &\to -E_{i+1,i} \\ \alpha_i^\vee &\to E_{ii} - E_{i+1,i+1} \end{aligned}$$

We can describe further relations that hold in finite-dimensional simple Lie algebras. To describe them, we pick a specific basis $\{e_\alpha, h_\alpha \mid \alpha \in \Phi\}$ of $\mathfrak{g}$ called Chevalley basis. This basis is chosen in particular such that we have $\mathfrak{g}_\alpha = \mathbb{C}e_\alpha$ and $h_\alpha = \alpha^\vee$. We then get the following proposition.

Proposition 3.1.8 ([Car89, Chapter 4.2])**.** Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with Chevalley basis $\{e_\alpha, h_\alpha \mid \alpha \in \Phi\}$. Then the following relations hold:

$$\begin{aligned} [h_\alpha, h_\beta] &= 0 \text{ for all } \alpha, \beta \in \Pi; \\ [h_\alpha, e_\beta] &= C_{\alpha,\beta} e_\beta \text{ for all } \alpha \in \Pi, \beta \in \Phi; \\ [e_\alpha, e_\beta] &= \begin{cases} 0 & \text{if } \alpha + \beta \notin \Phi \\ N_{\alpha,\beta} e_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \end{cases} \text{ for all } \alpha, \beta \in \Phi, \alpha \neq -\beta; \\ [e_\alpha, e_{-\alpha}] &= h_\alpha \text{ for all } \alpha \in \Phi. \end{aligned}$$

Here $C_{\alpha,\beta}$ and $N_{\alpha,\beta}$ are integer constants given in the following way: $C_{\alpha,\alpha} = 2$, and, for $\alpha \neq -\beta$, let $p, q \in \mathbb{N}$ be such that for all $-p \leq i \leq q$ we have $i\alpha + \beta \in \Phi$ but $-(p+1)\alpha + \beta \notin \Phi, (q+1)\alpha + \beta \notin \Phi$. Then $C_{\alpha,\beta} = \beta(\alpha^\vee) = p - q$ and $N_{\alpha,\beta} = \pm(p+1)$ where the sign depends on the explicit choice of e_α, e_β. It has been shown that $p, q \leq 3$ and hence $|C_{\alpha,\beta}| \leq 3, |N_{\alpha,\beta}| \leq 4$.

Remark 3.1.9. The finite-dimensional simple Lie algebras over $\mathbb{C}$ have been classified via their root system/Cartan matrix. The list of all possible Cartan matrices can be found in Figure 3.3, described via their Dynkin diagrams which will be defined below. A complete list with detailed and concrete descriptions of the root spaces and the algebras can be found in [Car89].

3.2 Chevalley and Steinberg groups

Given a finite-dimensional simple Lie algebra $\mathfrak{g}$ or equivalently a Cartan matrix, one can associate a subgroup of the automorphism group of $\mathfrak{g}$ to it. The idea is to exponentiate the nilpotent operators $\operatorname{ad} e_\alpha$:

$$x_\alpha(\lambda) := \exp(\lambda \operatorname{ad} e_\alpha) = \sum_{n\geq 0} \frac{\lambda^n (\operatorname{ad} e_\alpha)^n}{n!}.$$

Note that the sum is always finite and hence well defined. The adjoint Chevalley group of type A over $\mathbb{C}$ is then the group $G_A^{\mathrm{ad}} = \langle x_\alpha(\lambda) \mid \lambda \in \mathbb{C}, \alpha \in \Phi \rangle \leq \mathrm{GL}(\mathfrak{g}_A)$.

Motivated by the goal to classify all finite simple groups, this construction was adapted to work for all finite fields, and even more generally for arbitrary commutative rings. We will not go into further details here but refer to [Car89] for the classical and [SGA3] for a scheme theoretic/functorial approach.

The following definition of Chevalley groups by giving a presentation will be sufficient for this work.

Definition 3.2.1. Corresponding to an irreducible Cartan matrix A with root system Φ of rank at least 2, and a field K, there is an associated universal (or simply connected) Chevalley group, denoted $\mathrm{Chev}_A(K)$. Abstractly, it is generated by symbols $x_\alpha(s)$ for $\alpha \in \Phi$ and $s \in K$, subject to the relations

$$\begin{aligned} x_\alpha(s)x_\alpha(u) &= x_\alpha(s+u), \\ [x_\alpha(s), x_\beta(u)] &= \prod_{i,j>0} x_{i\alpha+j\beta}\left(C_{ij}^{\alpha,\beta} s^i u^j\right) \quad (\text{for } \alpha+\beta \neq 0), \\ h_\alpha(s)h_\alpha(u) &= h_\alpha(su) \quad (\text{for } s,u \neq 0), \\ \text{where} \quad h_\alpha(s) &= n_\alpha(s)n_\alpha(-1) \\ \text{and} \quad n_\alpha(s) &= x_\alpha(s)x_{-\alpha}\left(-s^{-1}\right)x_\alpha(s). \end{aligned}$$

Note that the $C_{ij}^{\alpha,\beta}$ are integers called structure constants that can be found in [Car89].

Remark 3.2.2. (i) Let K be a field and let $K[t]$ denote the polynomial ring in one variable over K. The simply-connected Chevalley group $\mathrm{Chev}_A(K[t])$ is, similar to the case of a Chevalley group over a field, generated by elements $x_\alpha(s)$ for $\alpha \in \Phi, s \in K[t]$ that satisfy the relations above, where for the third relation we have to add the extra assumption that u, s are invertible in $K[t]$. This can be found e.g. in [Reh75].

(ii) More generally, one can define Chevalley groups by introducing the *Chevalley–Demazure group scheme*, which is a functor from the category of commutative unital rings to the category of groups. Over fields and their polynomial ring in one variable these definitions yield the same groups.

Example 3.2.3. Corresponding to the Cartan matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ is the functor SL_4. This holds more generally for the Cartan matrix of $\mathrm{sl}_n(\mathbb{C})$ and SL_n.

The group $\mathrm{Chev}_A(K)$ acts via the adjoint representation on the Lie algebra $\mathfrak{g}_A$, now seen as a Lie algebra over K. For this, we start with the complex Lie algebra $\mathfrak{g}_A$, construct a $\mathbb{Z}$-form for it $(\mathfrak{g}_A)_\mathbb{Z}$ and then take the tensor product as abelian groups with K to get $(\mathfrak{g}_A)_K$, see [Car89, Chapter 4.4] for more details.

The adjoint representation $\mathrm{Ad} : \mathrm{Chev}_A(K) \to \mathrm{GL}((\mathfrak{g}_A)_K)$ has the following concrete realization [Car89, Chapter 4.4]:

$$(\mathrm{Ad}\, x_\alpha(s))(e_\beta) = \sum_{j\geq 0, \beta+j\alpha\in\Phi} M_{\alpha,\beta,j} s^j e_{\beta+j\alpha} \qquad \text{for } \alpha,\beta\in\Phi, \alpha\neq-\beta, s\in K \tag{3.5}$$

$$(\mathrm{Ad}\, x_\alpha(s))(e_{-\alpha}) = e_{-\alpha} - s h_\alpha + s^2 e_\alpha \qquad \text{for } \alpha\in\Phi, s\in K \tag{3.6}$$

$$(\mathrm{Ad}\, x_\alpha(s))(h) = h - \alpha(h) s e_\alpha \qquad \text{for } \alpha\in\Phi, h\in\mathfrak{h}, s\in K. \tag{3.7}$$

Here the $M_{\alpha,\beta,j}$ are constants given by the structure constants (from the commutator relations in the Chevalley group) $C_{i,j}^{\alpha,\beta}$ in the following way: $M_{\alpha,\beta,j} = C_{j,1}^{\alpha,\beta}$ for $j \geq 1$ and $M_{\alpha,\beta,0} = 1$.

Chevalley groups are closely related to Steinberg groups. Steinberg groups are given by a similar but simpler presentation which sometimes makes it easier to work with them. We will use Steinberg groups in Section 6.4 to describe the cohomology of the congruence KMS complexes.

Definition 3.2.4. Given an irreducible, spherical Cartan matrix A with root system Φ of rank at least 2, and a ring R, we define the Steinberg group of type A over R to be the group given by the following presentation.

$$\mathrm{St}_A(R) := \langle x_\alpha(u); \alpha\in\Phi, u\in R \mid \mathcal{R}\rangle$$

where the relations $\mathcal{R}$ are

$$x_\alpha(s)x_\alpha(u) = x_\alpha(s+u) \qquad \text{for } \alpha\in\Phi, u,s\in R,$$

$$[x_\alpha(s), x_\beta(u)] = \prod_{i,j>0} x_{i\alpha+j\beta}\left(C_{ij}^{\alpha,\beta} s^i u^j\right) \qquad \text{for } \alpha,\beta\in\ \Phi, \alpha+\beta\neq 0, s,u\in R^\times.$$

Remark 3.2.5. Note that all the relations of the Steinberg group also hold in the corresponding Chevalley group, while the Chevalley group has potentially more relations.

Over finite fields the Chevalley group and Steinberg group coincide, see e.g. [Wei13, Corollary 6.1.1]. Over rings, the kernel of the surjection from the Steinberg to the Chevalley group is the main object in K_2-theory.

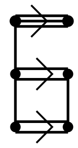


Figure 3.2: An example of a Dynkin diagram.

3.3 Generalized Cartan matrices and Kac–Moody algebras

In this section we generalize the definition of the Cartan matrix which will lead to infinite dimensional Lie algebras.

Definition 3.3.1. A *generalized Cartan matrix* (GCM) of rank n, $A = (A_{ij})_{i,j=1}^n$, is a matrix in $\mathrm{Mat}_n(\mathbb{Z})$ such that

(i) $A_{ii} = 2$ for all $1 \leq i \leq n$,

(ii) $A_{ij} \leq 0$ for all $1 \leq i \neq j \leq n$,

(iii) $A_{ij} = 0 \iff A_{ji} = 0$.

We can visualize a GCM using a labeled graph called a Dynkin diagram. Let $I = \{1, \ldots, n\}$ be the vertex set. We add an edge between vertices $i, j \in I$ according to the following rule:

- if $A_{ij} = 0$: i, j are not adjacent,
- if $|A_{ij}|= |A_{ji}|= 1$: i, j are connected by a single edge,
- if $|A_{ij}|= 2, |A_{ji}|= 1$: i, j are connected by a double edge equipped with an arrow pointing towards i,
- if $|A_{ij}|= |A_{ji}|= 2$: i, j are connected by a double edge with arrows pointing in both directions,
- if $|A_{ij}|= 3, |A_{ji}|= 1$: i, j are connected by a triple edge with an arrow pointing towards i,
- in all other cases, we connect i, j by a bold-faced edge equipped with the ordered pair of integers $|A_{ij}|, |A_{ji}|$.

Example 3.3.2. In Figure 3.2 we show an example of a Dynkin diagram. The Dynkin diagrams describing Cartan matrices are less involved and can be seen in Figure 3.3.

We call a GCM irreducible if the corresponding Dynkin diagram is a connected graph.

Definition 3.3.3. Given a GCM A we define the (derived) *Kac–Moody* algebra corresponding to A, $\mathfrak{g}_A$ as the Lie algebra with presentation as in Definition 3.1.5.

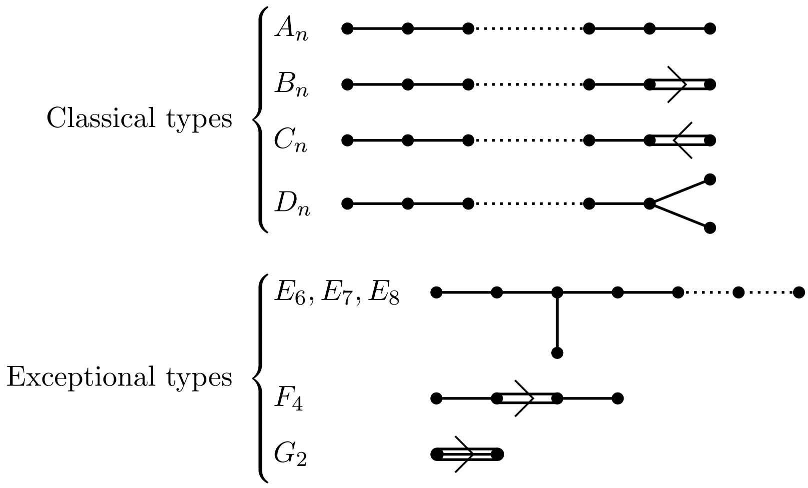


Figure 3.3: The Dynkin diagrams of irreducible spherical root systems

Let $A = (A_{ij})_{i,j \in I}$ be a GCM and $\mathfrak{h}$ a $\mathbb{C}$-vector space of dimension $|I| + \operatorname{corank}(A)$. Let $\Pi = \{\alpha_i \mid i \in I\}, \Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$ be linear independent subsets of $\mathfrak{h}^*, \mathfrak{h}$ respectively such that $\alpha_j(\alpha_i^\vee) = A_{ij}$.

We want to obtain a root space decomposition for $\mathfrak{g}_A$ similar to the classical case. If $\det A = 0$ then our choice of $\mathfrak{h}$ will be too large. Thus we set $\mathfrak{h}' = \sum_{i \in I} \mathbb{C}\alpha_i^\vee$ and then get

$$\mathfrak{g}_A = \mathfrak{h}' \oplus \bigoplus_{\alpha \in \Theta} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}_A \mid \forall f \in \mathfrak{h} : [f, x] = \alpha(f)x\}$ and $\Theta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq \{0\}\}$. We call Θ the root system associated to A. The set Π forms again a root basis which partitions the roots into positive and negative ones $\Theta^\pm = \{\pm \sum_{i \in I} \mathbb{N}\alpha_i\} \cap \Theta$.

We define the reflections s_α for $\alpha \in \Theta$ as before and define the *Weyl group*

$$W = \langle s_\alpha \mid \alpha \in \Theta \rangle \leq \mathrm{GL}(\mathfrak{h}^*).$$

We recall a few differences between the case of starting from a GCM and a classical Cartan matrix.

(i) When considering a GCM, the associated root system and Weyl group can be infinite. In fact a GCM A is a classical Cartan matrix if and only if the associated root system (and hence also Weyl group) is finite.

(ii) In case the root system is infinite, $\mathfrak{g}_A$ will be infinite dimensional, but each root space $\mathfrak{g}_\alpha$ is finite-dimensional although not necessarily 1-dimensional.

(iii) The root system splits into a real and an imaginary part. The real roots are those in the orbit of the simple roots under the action of the Weyl group, i.e. $\Theta^{\mathrm{re}} = W.\Pi$

and all other roots are called imaginary, $\Theta^{\mathrm{im}} = \Theta \setminus \Theta^{\mathrm{re}}$. If A is a Cartan matrix, then all roots are real. Later, we will mostly care about the real roots and write Φ for Θ^{re}.

(iv) For $\alpha \in \Theta^{\mathrm{re}}$ we have $\dim(\mathfrak{g}_\alpha) = 1$. In particular $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i$ and hence for each $\alpha = w.\alpha_i \in \Theta^{\mathrm{re}}$ we get an element e_α such that $\mathfrak{g}_\alpha = \mathbb{C}e_\alpha$.

(v) If $\alpha \in \Theta^{\mathrm{re}}, x \in \mathfrak{g}_\alpha$ then $\operatorname{ad} x \in \operatorname{End}(\mathfrak{g}_A)$ is locally nilpotent, i.e. for every $y \in \mathfrak{g}_A$ there exists an n such that $(\operatorname{ad} x)^n y = 0$.

Proposition 3.3.4 ([Mar18, Prop. 5.3, Def. 5.4, Cor. 5.5])**.** Let $A \in \operatorname{Mat}_n(\mathbb{Z})$ be an irreducible GCM. Given a vector $v \in \mathbb{R}^n$, we write $v > 0$ if all entries of v are > 0 and similarly for ≥ 0. Then exactly one of the following holds:

(i) There exists $u > 0$ such that $Au > 0$. In this case $\det A \neq 0$, and we call A *of finite type or spherical.*

(ii) There exists $u > 0$ such that $Au = 0$. In this case $\operatorname{corank} A = 1$, and we call A *of affine type.*

(iii) There exists $u > 0$ such that $Au < 0$. In this case we call A *of indefinite type.*

Remark 3.3.5. (i) The GCMs of finite type are precisely the classical Cartan matrices, classified via their Dynkin diagrams in Figure 3.3.

(ii) The affine GCMs have also been classified. The affine GCMs of *untwisted* type are those depicted in Figure 3.4. There are further affine cases that arise from a twist by a diagram automorphism. They can be seen e.g. in [Mar18, Table 5.1].

Definition 3.3.6. Let A be a generalized Cartan matrix with index set $I = \{1, \dots, n\}$, and let $J \subseteq I$.

(i) The subset J is called *spherical* if $A_J = (A_{ij})_{i,j\in J}$ is spherical. Given $\ell \geq 2$, A is *ℓ-spherical* if every subset $J \subseteq I$ of size ℓ is spherical.

(ii) We denote by Q_A the set of spherical subsets of I associated to the generalized Cartan matrix A.

(iii) A generalized Cartan matrix A is *purely ℓ-spherical* if every spherical subset $J \subseteq I$ is contained in a spherical subset of size ℓ. (In particular, no set of size $\ell + 1$ is spherical for a purely ℓ-spherical generalized Cartan matrix.)

(iv) Analogously, we call a GCM *ℓ-classical* if each irreducible factor of $A_J = (A_{ij})_{i,j\in J}$ is of classical type A_k, B_k, C_k, D_k (see Figure 3.3) for each $J \subseteq I$ with $|J| \leq \ell$.

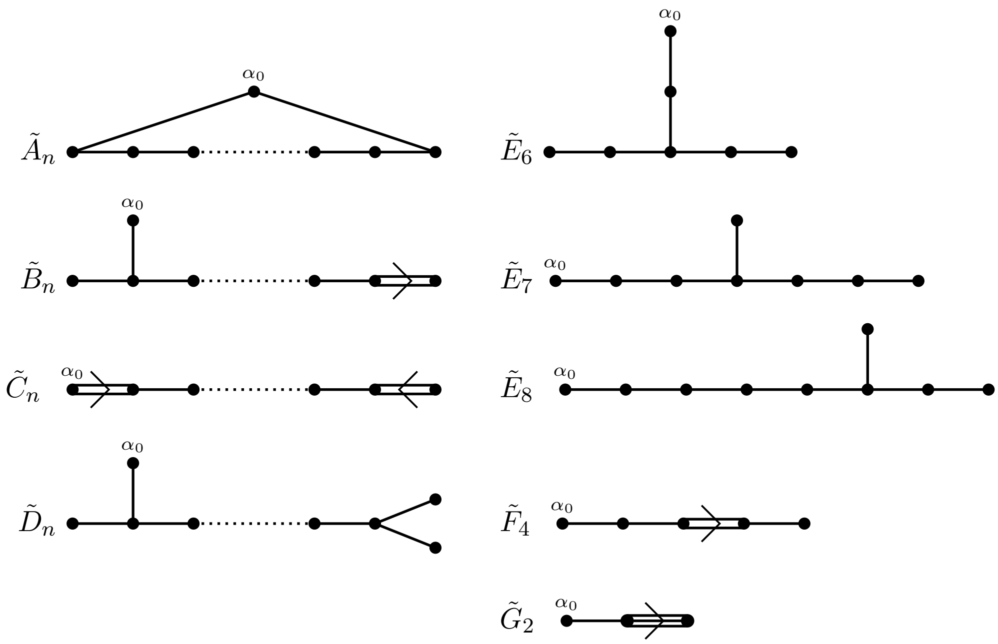
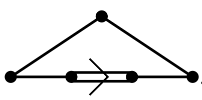


Figure 3.4: The Dynkin diagrams of irreducible untwisted affine GCMs.

Remark 3.3.7. All rank $n+1$, n-spherical diagrams have been classified and are precisely the affine diagrams and the so called compact hyperbolic diagrams. The latter have rank at most 5, where the diagram of rank 5 is exactly the one that we have to exclude to get n-classicality. See e.g. [Hum90, Chapter 6.9] for the classification result.

The non-spherical, rank $n+1$, n-classical diagrams, with $n \geq 3$ are precisely the non-spherical rank $n+1$ diagrams which are n-spherical but not one of the following

- affine diagrams $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4, \tilde{G}_2$,
- twisted affine diagrams $E_6^{(2)}, D_4^{(3)}$ (in the notation of [Mar18, Table 5.1])
- compact hyperbolic diagram .

The affine case In this section we have a closer look at the untwisted affine case. We describe the root system of type $\tilde{X}_n$ ($X = A, B, C$ etc.) with respect to the spherical root system of type X_n. This will later allow us to describe the untwisted affine Kac–Moody groups using the corresponding Chevalley groups.

Let $\mathring{A} = (A_{i,j})_{i,j\in\mathring{I}}, \mathring{I} = \{1, \dots n\}$ be an irreducible (spherical) Cartan matrix with spherical root system $\mathring{\Phi}$ and set of simple roots $\mathring{\Pi} = \{\alpha_1, \dots, \alpha_n\}$. This root system has a unique highest root $\gamma \in \mathring{\Phi}$, which means that γ is such that for any $\alpha_i \in \mathring{\Pi}$ we have $\gamma + \alpha_i \notin \mathring{\Phi}$.

We set $\alpha_0 = -\gamma$ and $I = \{0, \ldots, n\}$. The idea now is to consider the set $\Pi = \{\overline{\alpha}_0, \ldots, \overline{\alpha}_n\}$ as the set of simple roots of a root system Θ such that for all $i, j \in I$ the roots $\overline{\alpha}_i, \overline{\alpha}_j$ generate the same rank two subsystem as α_i, α_j in $\mathring{\Phi}$. The resulting root systems Θ with simple roots $\Pi = \{\overline{\alpha}_0, \ldots, \overline{\alpha}_d\}$ and generalized Cartan matrix A are described by the Dynkin diagrams in Figure 3.4 and are precisely the ones of affine (untwisted) type, see [Mar18, Theorem 5.21] for a proof. Note that A is of rank $n+1$, n-spherical and purely n-spherical.

For $\alpha = \sum_{i=1}^{n} \lambda_i \alpha_i \in \mathring{\Phi}$ we set $\overline{\alpha} = \sum_{i=1}^{n} \lambda_i \overline{\alpha}_i$ for the corresponding root in Θ. We can describe all roots in Θ. Let $(d_0, \ldots, d_n) \in \mathbb{Z}^{n+1}$ be such that it spans the kernel of A and with $\gcd(d_0, \ldots, d_n) = 1$. This exists since A has integral entries and corank 1. Let $\delta = \sum_{i=0}^{n} d_i \alpha_i$. We have $\bar{\delta} = \sum_{i=0}^{n} d_i \overline{\alpha}_i = \overline{\alpha}_0 + \bar{\gamma}$, see [Mar18, Remark 5.11]. Furthermore,

$$\Theta = \left\{ \overline{\alpha} + m\bar{\delta} \mid \alpha \in \mathring{\Phi}, m \in \mathbb{Z} \right\} \cup \left\{ m\bar{\delta} \mid m \in \mathbb{Z}, m \neq 0 \right\}$$

where the first set in the union is the set of all real roots, and the second describes all imaginary roots. We will use the notation $a_{\alpha,m} = \overline{\alpha} + m\bar{\delta}$ to describe the real roots. Note that $\overline{\alpha}_0 = a_{-\gamma,1}$ and that a root $a_{\alpha,m}$ is positive if and only if $m \geq 1$ or $m = 0$ and $\alpha \in \mathring{\Phi}^+$ (see e.g. [Mar18, proof of Theorem 7.90]).

3.4 Minimal Kac–Moody groups

We want to generalize the construction of Chevalley groups from finite-dimensional Lie algebras to Kac–Moody algebras. A first idea would be to start with a GCM A and the associated complex Lie algebra $\mathfrak{g}_A$ and again try to exponentiate $\operatorname{ad} x$ for $x \in \mathfrak{g}_A$. The problem is, that for $x \in \mathfrak{g}_\alpha \setminus \{0\}$ for some $\alpha \in \Theta$ the endomorphism $\operatorname{ad} x$ is only locally nilpotent and hence the exponential well defined if α is a real root. In particular, in this case the $\mathfrak{g}_\alpha$ are 1-dimensional and we can find a basis vectors e_α of $\mathfrak{g}_\alpha$. Restricting to only real roots leads to the following definition.

Definition 3.4.1. Let A be a GCM. Then the *minimal adjoint Kac–Moody group* over $\mathbb{C}$ of type A is

$$G_A^{\mathrm{ad}} = \langle \exp(\lambda \operatorname{ad} e_\alpha) \mid \lambda \in \mathbb{C}, \alpha \in \Theta^{\mathrm{re}} \rangle \leq \mathrm{GL}(\mathfrak{g}_A).$$

Similar to the finite-dimensional case, we again want to broaden the definition to every field and even more general to obtain a functor from the category of commutative unital rings to the category of groups. This will be the *constructive Tits functor*. We will not elaborate more on how this definition was derived, but simply state it and refer to [Mar18] for more information and proofs. Before defining the functor, we need one preliminary definition.

Definition 3.4.2. (i) A pair of real roots $\alpha, \beta \in \Theta^{\mathrm{re}}$ is called *prenilpotent* if there exist u, v in the Weyl group W such that $u.\alpha, u.\beta \in \Theta^+$ and $v.\alpha, v.\beta \in \Theta^-$.

(ii) For a pair of roots α, β we set

$$]\alpha,\beta[_{\mathbb{N}}= \{i\alpha + j\beta \mid i,j \in \mathbb{N}_{>0}\} \cap \Theta;\ [\alpha,\beta]_{\mathbb{N}} =]\alpha,\beta[_{\mathbb{N}} \cup \{\alpha,\beta\}.$$

Remark 3.4.3. If the roots α, β are prenilpotent, then $]\alpha,\beta[_{\mathbb{N}}$ is finite and only contains real roots.

Definition 3.4.4 ([Mar18, Definition 7.47])**.** Let A be a GCM, and K a commutative unital ring. Then the constructive Tits functor $G_A(K)$ of type A evaluated at K is given by the following group presentation

$$G_A(K) = \langle x_\alpha(\lambda), r^{\alpha_i^\vee}; \alpha \in \Theta^{\mathrm{re}}, \lambda \in K, i \in I, r \in K^\times \mid \mathcal{R}\rangle.$$

We denote $\tilde{s}_i(r) := x_{\alpha_i}(r)x_{-\alpha_i}(r^{-1})x_{\alpha_i}(r)$ for $i \in I, r \in K^\times$ and $\tilde{s}_i = \tilde{s}_i(1)$.

The set of relations $\mathcal{R}$ consists of the following.

(i) For all $\alpha \in \Theta^{\mathrm{re}}, \lambda, \mu \in K$: $x_\alpha(\lambda)x_\alpha(\mu) = x_\alpha(\lambda + \mu)$. In particular, the subgroup $U_\alpha = \langle x_\alpha(\lambda); \lambda \in K\rangle$ is isomorphic to $(K, +)$. These subgroups are called *root subgroups.*

(ii) For all $i \in I, r, s \in K^\times$: $r^{\alpha_i^\vee} s^{\alpha_i^\vee} = (rs)^{\alpha_i^\vee}$ and for all $i, j \in I, r, s \in K^\times$: $r^{\alpha_i^\vee} s^{\alpha_j^\vee} = s^{\alpha_j^\vee} r^{\alpha_i^\vee}$. In particular, the subgroup $T_K = \langle r^{\alpha_i^\vee}; r \in K^\times, i \in I\rangle$ is isomorphic to $((K^\times)^{|I|}, \cdot)$. This subgroup is called the *torus* of $G_A(K)$.

(iii) For all $\alpha, \beta \in \Theta^{\mathrm{re}}$ prenilpotent, $t, u \in K$:

$$[x_\alpha(t), x_\beta(u)] = \prod_{\gamma = i\alpha + j\beta \in]\alpha,\beta[_{\mathbb{N}}} x_\gamma(C^{\alpha\beta}_{i,j} t^i u^j)$$

where $C^{\alpha\beta}_{ij}$ are integer structure constants similar to the ones in the Chevalley group definition. They have been explicitly calculated in [Mor87].

(iv) For all $r \in K^\times, t \in K, i, j \in I$:

$$r^{\alpha_i^\vee} x_{\alpha_j}(t) r^{-\alpha_i^\vee} = x_{\alpha_j}(r^{A_{ij}}t).$$

(v) For all $r \in K^\times, i, j \in I$:

$$\tilde{s}_i r^{\alpha_j^\vee} \tilde{s}_i^{-1} = r^{\alpha_j^\vee} (r^{-A_{ji}})^{\alpha_i^\vee} =: r^{s_i \alpha_j^\vee}.$$

(vi) For all $r \in K^\times, i \in I$:

$$\tilde{s}_i(r^{-1}) = \tilde{s}_i r^{\alpha_i^\vee}.$$

(vii) For all $\gamma \in \Theta^{\mathrm{re}}, t \in K, i \in I$:

$$\tilde{s}_i x_\gamma(t) \tilde{s}_i^{-1} = x_{s_i\gamma}(\pm t).$$

The sign $\pm$ in $x_{s_i\gamma}(\pm t)$ depends on the choice of the Chevalley basis vector e_α.

We then define the *minimal, split* Kac–Moody group of type A over the field K as $G_A(K)$.

Remark 3.4.5. (i) The notation $r^{\alpha_i^\vee}$ as exponent is just a formal notation to hint at the multiplicative behavior.

(ii) Relation (iii) is called the Steinberg relation.

(iii) Relation (iv) describes how the torus normalizes the root subgroups.

(iv) Relation (v) describes how the $\bar{s}_i$ normalize the torus. The expression $s_i\alpha_j^\vee$ is the image of $\alpha_j^\vee$ under the reflection associated to $\alpha_i^\vee$.

(v) Relation (vi) implies that the torus is generated by the root subgroups.

(vi) Relation (vii) describes how action of the Weyl group elements s_i on the roots corresponds to conjugation by the $\tilde{s}_i$.

Remark 3.4.6. It is sensible to call this a generalization of the definition of Chevalley groups in the sense that if A is of finite type, then $G_A(K) \cong \mathrm{Chev}_A(K)$. Furthermore, we have a surjection from $G_A(\mathbb{C})$ to G_A^{ad} by mapping $x_\alpha(\lambda)$ to $\exp(\mathrm{ad}(\lambda e_\alpha))$ and $(e^\lambda)^{\alpha_i^\vee}$ to $\exp(\mathrm{ad}(\lambda\alpha_i^\vee))$.

For most Kac–Moody groups it is not known if they can be described explicitly e.g. via matrix groups. For the affine types however this is possible and relates them to the so called loop groups. We have a closer look at the untwisted affine case.

Let $\mathring{A}$ be a Cartan matrix with root system $\mathring{\Phi}$ and set of simple roots $\mathring{\Pi} = \{\alpha_1, \ldots, \alpha_n\}$ and let A be the corresponding untwisted affine GCM with root system Θ and simple roots $\Pi = \{\overline{\alpha}_0, \ldots, \overline{\alpha}_n\}$. Recall that we use the notation $a_{\alpha,m} = \overline{\alpha} + m\overline{\delta}$ to describe the real roots. Let K be a field. We have $G_A(K) \cong \mathrm{Chev}_{\mathring{A}}(K[t,t^{-1}])$ "up to torus elements". To get a precise isomorphism, we need to consider a central extension of $\mathrm{Chev}_{\mathring{A}}(K[t,t^{-1}])$ by $K^\times$. See [Mar18, Chapter 7.6] and reference therein for more details.

We describe the isomorphism between the Kac–Moody group and the Chevalley group restricted to the subgroup U^+ (which we define next), in more detail. The subgroup U^+ will play a crucial role later in the discussion of the KMS groups.

Definition 3.4.7. Let A be a GCM, K a field. We denote by $U_A^+(K)$ the positive maximal unipotent subgroup of $G_A(K)$ i.e.

$$U_A^+(K) = \langle x_\alpha(\lambda); \alpha \in \Theta^{\mathrm{re}} \cap \Theta^+, \lambda \in K \rangle \leq G_A(K).$$

If A, K are clear from the context we also write $U^+ = U_A^+(K)$.

Proposition 3.4.8 ([Mar18, Theorem 7.90])**.** Let $\mathring{A}$ be a Cartan matrix with corresponding untwisted affine GCM A, K a field. Then the following is a well-defined injective homomorphism.

$$\begin{aligned}\varphi : U_A^+(K) &\to \mathrm{Chev}_{\mathring{A}}(K[t])\\ x_{a_{\alpha,m}}(\lambda) &\mapsto u_\alpha(\lambda t^m).\end{aligned}$$

For clarity we denoted the generators of the Chevalley group by $u_\alpha(\lambda)$.

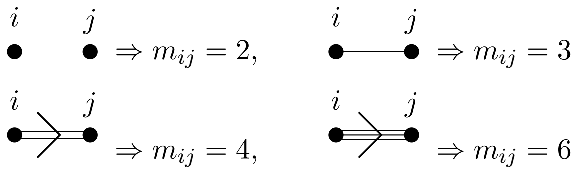
3.5 (Twin) buildings

In this section, we introduce Coxeter complexes, the building blocks of buildings, we define what a building is and show how it can be constructed from a group. We then describe buildings and more generally partite simplicial complexes from another viewpoint via chamber systems. We discuss opposition in spherical buildings and introduce twin buildings in order to be able to talk about opposition also in the non-spherical case. We then describe how Chevalley and Kac–Moody groups give rise to (twin) buildings.

Coxeter complex To each Dynkin diagram with vertex set I we can associate a group, called Coxeter group, in the following way:

$$W = \langle s_i; i \in I \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1; i, j \in I, i \neq j\rangle \tag{3.8}$$

where



and in all other cases $m_{ij} = \infty$, which means that there is no relation between s_i and s_j. Note that m_{ij} is defined to be symmetric in i, j, this implies that the root systems of type B_n and C_n give rise to the same Coxeter group.

The Coxeter group is isomorphic to the Weyl group generated by the reflections along the roots of the Lie algebra $\mathfrak{g}_A$ associated to the same Dynkin diagram, see e.g. [Mar18, Proposition 4.22].

Remark 3.5.1. In general, a Coxeter group can be defined for arbitrary values m_{ij} satisfying $m_{ii} = 1$ and $m_{ij} = m_{ji} \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$ for $i \neq j$. These are usually described by Coxeter diagrams which are roughly speaking unoriented versions of Dynkin diagrams. But the Coxeter group is finite if and only if the values m_{ij} come from a spherical Dynkin diagram, expect for one infinite family with two generators and two exceptional types, which will not give rise to buildings. In this work, we are only interested in those coming from Dynkin diagrams.

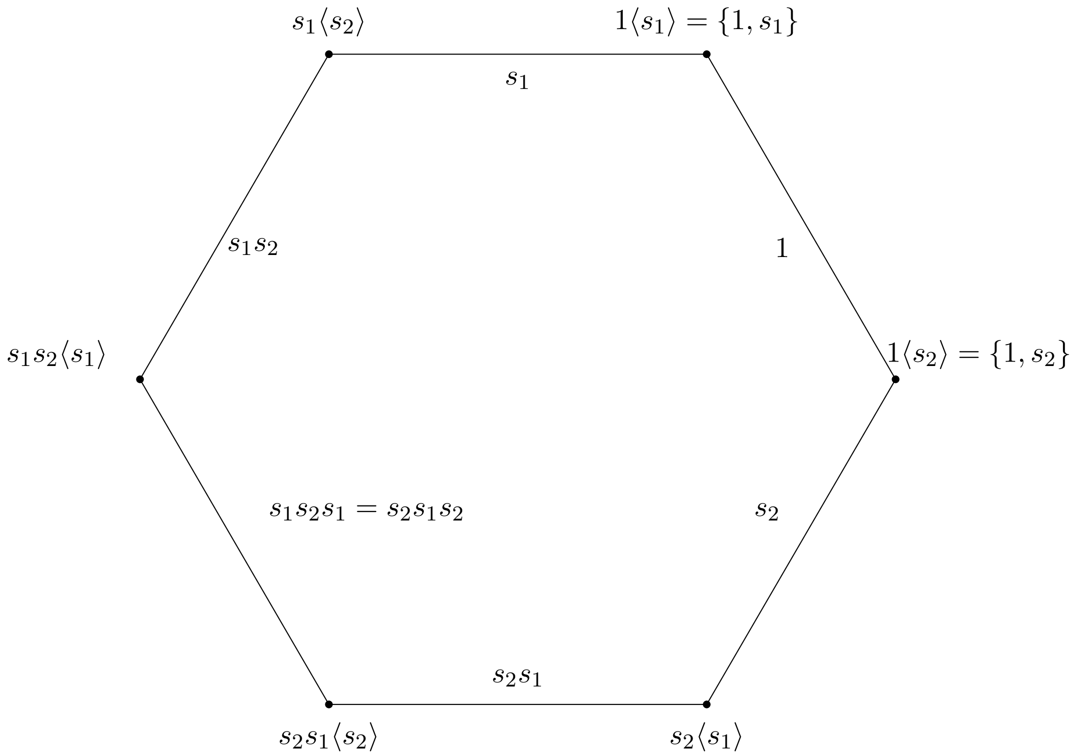


Figure 3.5: Coxeter complex of type A_2. The labels show the corresponding elements of the coset complex.

A *Coxeter system* is a pair (W, S) consisting of a Coxeter group W together with a set of generators $S = \{s_i \mid i \in I\}$ such that W has a presentation as in Equation (3.8) with respect to S. We define $W_J = \langle s_j \mid j \in J\rangle \leq W$ for all $J \subseteq I$. To W, we associate a *Coxeter complex* which can be viewed as the coset complex

$$\mathcal{CC}(W, (W_{I\setminus\{i\}})_{i\in I}).$$

This complex has a wide range of geometrical interpretation which we will not discuss here. More details can be found e.g. in [AB08, Chapter 3].

Example 3.5.2. (i) For type A_2 we have

$$W = \langle s_1, s_2 \mid s_i^2 = 1, s_1s_2s_1 = s_2s_1s_2\rangle.$$

The vertices of the Coxeter complex correspond to cosets $W/\langle s_1\rangle \sqcup W/\langle s_2\rangle$ and edges correspond to elements of W. The graph is shown in Figure 3.5. In general, the Coxeter group of type A_n is the symmetric group on $n+1$ elements.

(ii) The Coxeter complex of type $\tilde{A}_2$ can be visualized as the tiling of the plane by equilateral triangles. The Coxeter group has the form

$$W = \langle s_0, s_1, s_2 \mid s_i^2 = 1, (s_0s_1)^3 = (s_1s_2)^3 = (s_0s_2)^3 = 1\rangle$$

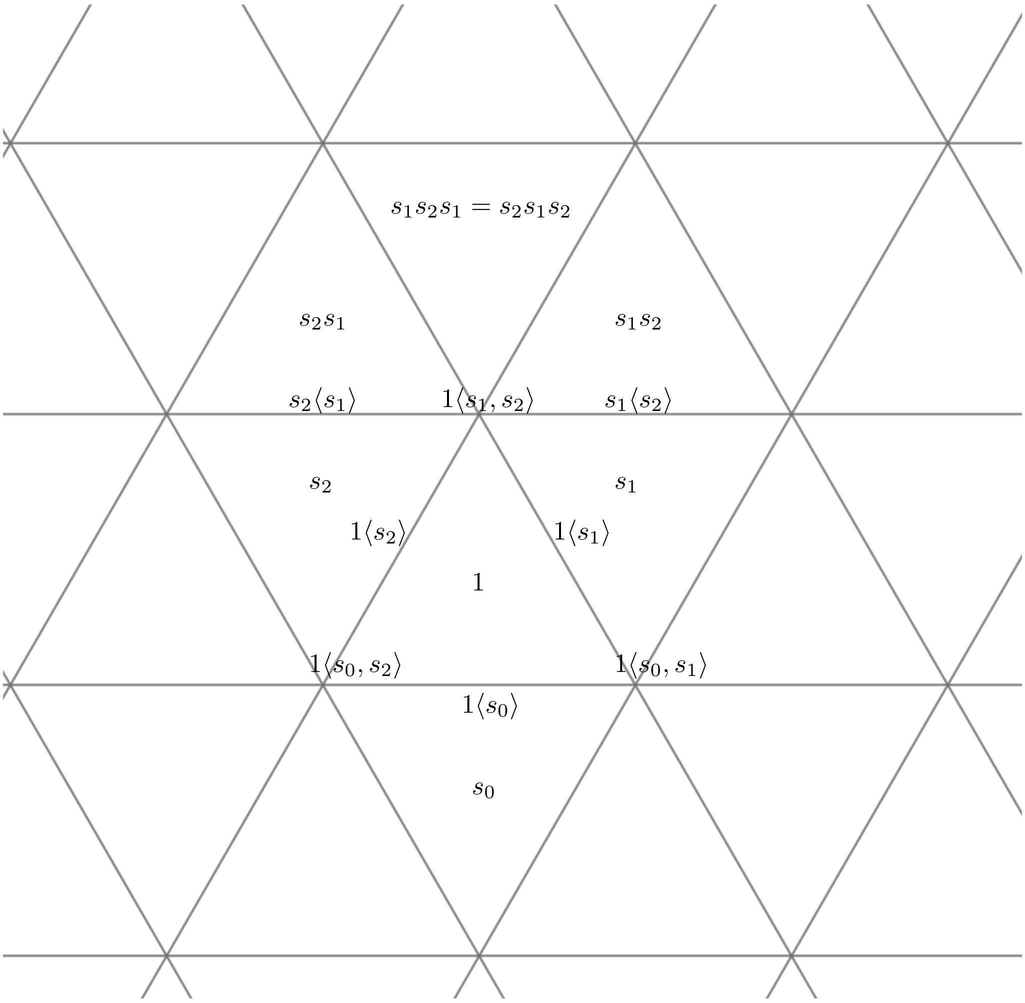


Figure 3.6: Coxeter complex of type $\tilde{A}_2$ with some labels from the corresponding coset complex.

and the Coxeter complex has vertex set $W/\langle s_0, s_1\rangle \sqcup W/\langle s_1, s_2\rangle \sqcup W/\langle s_0, s_2\rangle$, edge set $W/\langle s_0\rangle \sqcup W/\langle s_1\rangle \sqcup W/\langle s_2\rangle$ and set of triangles corresponding to W. It is depicted in Figure 3.6.

Buildings One way to define a building is as a simplicial complex with a family of subcomplexes called apartments. Each apartment is isomorphic to the same Coxeter complex. The type of the Coxeter complex, i.e. the underlying Coxeter diagram - an unoriented version of the Dynkin diagram, is also what we call the type of the building. Each two simplices of the building have to be contained in a common apartment and each two apartments are isomorphic with an isomorphism that acts as the identity on the intersection. The maximal simplices in a building are called *chambers*.

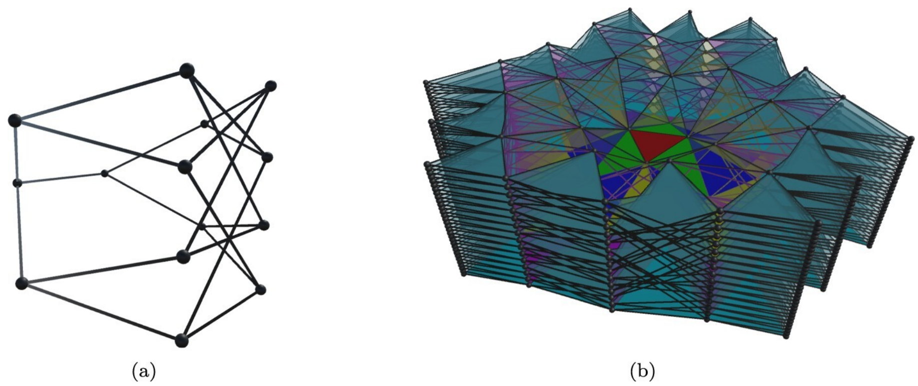


Figure 3.7: (a) shows the building of type A_2 of thickness 3, (b) shows the building of type $\tilde{A}_2$ of thickness 3 up to distance 5 from the chamber at the center. This is [BS22, Figure 2].

Two visualizations of (parts of) buildings can be seen in Figure 3.7. One can see the Coxeter complexes from Example 3.5.2 as substructures.

One way to construct buildings is via a group with a BN-pair (and for thick, irreducible spherical buildings of dimension 2 and larger, all buildings can be constructed that way, as was proven by Jacques Tits in [Tit74]).

Given a group G, a BN-pair of G is a pair of subgroups $B, N \leq G$ satisfying certain properties, see [AB08, Definition 6.55]. In particular, $G = BN$ and setting $T = B \cap N$ we have that T is normal in N and $W = N/T$ admits a set of generators $S = \{s_i \mid i \in I\}$, such that (W, S) is a Coxeter system. The group W is also called the Weyl group of the BN-pair. To describe the building, we need to define the parabolic subgroups $P_J := BW_JB$ for $J \subset I$ (here $W_J = \langle s_j \mid j \in J\rangle \leq W$ and while W is not contained in G this double coset is well defined since $T \subseteq B$ and hence any choice of a representative in N of an element in $W = N/T$ gives rise to the same double coset). The building can then be described as the following coset complex:

$$\Delta(G, B) := \mathcal{CC}(G; (P_{I\setminus\{j\}})_{j\in I}).$$

We recall some well-known facts about the parabolic subgroups.

Lemma 3.5.3. We have that $P_J = \langle P_{\{i\}} \mid i \in J\rangle$ and $P_J \cap P_L = P_{J\cap L}$ for all $J, L \subseteq I$.

Proof. The first statement is e.g. shown in [AB08, Propostion 6.36] and the second statement for example in [Sch95, Theorem 4.3.8]. □

Since $P_J \cap P_L = P_{J\cap L}$ for any $J, L \subseteq I$ we can view $\Delta(G, B)$ also as the poset $\{gP_J \mid g \in G, J \subseteq I\}$ ordered by reversed inclusion. In this case, the simplex $\{gP_{I\setminus\{j\}} \mid j \in J\} \in \mathcal{CC}(G; (P_{I\setminus\{j\}})_{j\in S})$ corresponds to $gP_{I\setminus J}$ for $J \subseteq I, g \in G$. In particular, the set of maximal simplices corresponds to G/B.

If the Weyl group is finite, we call the building spherical. This coincides with our definition of calling a GCM spherical/of finite type.

Chamber systems In this section, we introduce the notion of chamber system, which is another approach to describe partite complexes like the coset complexes and buildings. For the exposition we follow [Sch95].

Definition 3.5.4 ([Sch95, Definition 1.2.1])**.** A chamber system over some index set I is a set $\mathcal{C}$ together with equivalence relations $\sim_i$ on $\mathcal{C}$ for each $i \in I$. The elements of $\mathcal{C}$ are called chambers.

Similar to coset complexes, we can construct chamber systems using a group and a family of subgroups.

Proposition 3.5.5. [Sch95, Proposition 1.4.5] Let G be a group, $B \leq G$ a subgroup and $G_i, i \in I$ a family of subgroups of G containing B. Let $\mathcal{C}(G, B, G_i, i \in I)$ be the set G/B together with the family of relations

$$xB \sim_i yB \iff xG_i = yG_i.$$

Then $\mathcal{C}(G, B, G_i, i \in I)$ is a chamber system over I.

We can view a chamber system $(\mathcal{C}, \sim_i, i \in I)$ as a labeled graph with vertex set $\mathcal{C}$ in which two chambers C, D are connected by an edge with label i if $C \sim_i D$. This viewpoint gives rise to the following, motivated by the path metric on the labeled graph.

Definition 3.5.6. Let $(\mathcal{C}, \sim_i, i \in I)$ be a chamber system. A *gallery* in $\mathcal{C}$ is a chain $C_0 \sim_{i_1} C_1 \sim_{i_2} \ldots \sim_{i_n} C_n$ such that $C_k \in \mathcal{C}, i_k \in I$ for all k. We call $(i_1, \ldots, i_n)$ the type of the gallery.

The gallery is called minimal if its length n is minimal among all galleries between the two endpoints.

Remark 3.5.7 ([Sch95, Chapter 1.3])**.** Let G be a group, $H_i, i \in I$ a family of subgroups over a finite index set I. For $J \subseteq I$ set $H_J = \cap_{j \in J} H_j$. Assume that $\langle H_{I \setminus \{j\}} \mid j \in J\rangle = H_{I \setminus J}$ for all $J \subseteq I$. Then $\mathcal{CC}(G; (H_i)_{i \in I})$ corresponds to the chamber complex $\mathcal{C}(G, \cap_{i \in I} H_i, H_{I \setminus \{i\}}, i \in I)$ in a natural way: Given the coset complex $X = \mathcal{CC}(G; (H_i)_{i \in I})$ we consider as chambers $\mathcal{C}$ the maximal simplices of X which are of the form $\{gH_i \mid i \in I\}$. Note that two maximal simplices $\{gH_i \mid i \in I\}, \{hH_i \mid i \in I\}$ are equal if and only if $h^{-1}g \in \cap_{i \in I} H_i$. Hence the set of maximal simplices corresponds to G/H_I. We say that $\{gH_i \mid i \in I\} \sim_j \{hH_i \mid i \in I\}$ if $gH_i = hH_i$ for all $i \in I \setminus \{j\}$ which is equivalent to $gH_{I \setminus \{j\}} = hH_{I \setminus \{j\}}$.

On the other hand, let $\mathcal{C}(G, \cap_{i \in I} H_i, H_{I \setminus \{i\}}, i \in I)$ be a chamber complex. For $J \subseteq I$ we denote for two chambers $C, D \in \mathcal{C}$ that $C \sim_J D$ if and only if there is a gallery from C to D of type $(i_1, \ldots, i_n)$ such that $\{i_1, \ldots, i_n\}$ is a subset of J. For $C \in \mathcal{C}$ the J-residue of C is the set $R_J(C) = \{D \in \mathcal{C} \mid C \sim_J D\}$. In our case, we have

$$R_J(gH_I) = \{hH_I \mid h \in g\langle H_{I \setminus \{j\}} \mid j \in J\rangle\} = \{hH_I \mid h \in gH_{I \setminus J}\}.$$

In particular, $R_{I \setminus \{i\}}(gH_I) = \{hH_I \mid h \in gH_i\}$. We define the set of vertices $X(0) := \{(R_{I \setminus \{j\}}(gH_I), j) \mid j \in I, g \in G\}$ which by the above argument is isomorphic to $\sqcup_{j \in I} G/H_j$.

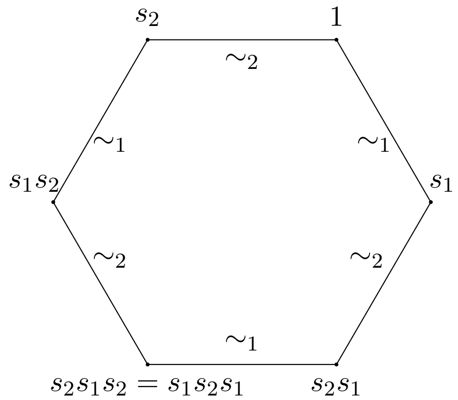


Figure 3.8: Coxeter complex of type A_2. The labels show the corresponding chambers and equivalence relations.

Furthermore, we say that two vertices $(R_{I\setminus\{j\}}(gH_I), j), (R_{I\setminus\{i\}}(hH_I), i)$ are adjacent if $i \neq j$ and $R_{I\setminus\{j\}}(gH_I) \cap R_{I\setminus\{i\}}(hH_I) \neq \emptyset$, where the latter is equivalent to $gH_j \cap hH_i \neq \emptyset$. From here, we build a simplicial complex by defining that a set of vertices forms a simplex if and only if all vertices are pairwise adjacent.

Viewing the Coxeter complex of a Coxeter system (W, S) as chamber system, we get $\mathcal{C}(W, \{1\}, \langle s\rangle, s \in S)$. A chamber system is called a building, if there exists a family of sub-chamber systems called apartments such that each apartment is isomorphic to the same Coxeter complex, each two chambers are contained in a common apartment and each two apartments are isomorphic with an isomorphism that acts as the identity on the intersection.

Note that this definition is compatible with Remark 3.5.7 and the two definitions of buildings coincide.

Example 3.5.8. The Coxeter complex of type A_2 has a set of chambers the full group W and two equivalence relations, as shown in Figure 3.8

Proposition 3.5.9 ([AB08, Proposition 4.81])**.** Let $(\mathcal{C}, \sim_i, i \in I)$ be a building of type (W, S) viewed as chamber system. There is a function $\delta : \mathcal{C} \times \mathcal{C} \to W$ with the following properties:

(i) Given a minimal gallery $\Gamma : C_0, \ldots, C_d$ of type $\mathbf{s}(\Gamma) := (s_1, \ldots, s_d)$, then $\delta\,(C_0, C_d)$ is the element $w = s_1 \cdots s_d$.

(ii) Let C and D be chambers, and let $w = \delta(C, D)$. The function $\Gamma \mapsto \mathbf{s}(\Gamma)$ gives a 1-1 correspondence between minimal galleries from C to D and reduced decompositions of w.

Definition 3.5.10. Let $(\mathcal{C}, \sim_i, i \in I)$ be a building of type (W, S) viewed as chamber system. The function $\delta : \mathcal{C} \times \mathcal{C} \to W$ of Proposition 3.5.9 is called the *Weyl distance function* associated to $\mathcal{C}$.

Remark 3.5.11. Another way to define buildings is via the so called W-metric approach. Given a Coxeter system (W, S), a building of type (W, S) is a set $\mathcal{C}$ together with a function $\delta : \mathcal{C} \times \mathcal{C} \to W$ satisfying certain axioms, see [AB08, Definition 5.1]. These definitions are equivalent, in particular the Weyl distance function from Definition 3.5.10 satisfies the axioms.

Let G be a group with BN-pair (B, N). We described the corresponding building as $\mathcal{CC}(G; (P_{I\setminus\{i\}})_{i\in I})$. By Lemma 3.5.3 and Remark 3.5.7 this can also be seen as the chamber system $\mathcal{C}(G, B, P_{\{i\}}, i \in I)$. We can describe the Weyl distance via the group structure: Recall that the Weyl group of the BN-pair is $W = N/(B \cap N)$. Thus, with slight abuse of notation, it is well defined to consider cosets and double cosets wB, BwB for $w \in W$. Furthermore, the map $C : W \to B\backslash G/B : w \mapsto BwB$ is a bijection. This is a consequence of the Bruhat decomposition of G, see [AB08, Chapter 6.2]. We can describe the Weyl distance on chambers of the building $\Delta(G, B)$ associated to the BN-pair as follows, for $g, h \in G, w \in W$:

$$\delta(gB, hB) = w \iff g^{-1}h \in BwB.$$

Similar to how a spherical (i.e. finite) root system has a highest root, a spherical Coxeter system (W, S) has a unique longest word w_0, where “longest” refers to the word metric with respect to S. We have $w_0^2 = 1$, see e.g. [AB08, Proposition 1.77].

Definition 3.5.12. Let $(\mathcal{C}, \sim_i, i \in I)$ be a building of type (W, S) viewed as chamber system, where (W, S) is spherical. Let w_0 denote the longest element of W. Then two chambers $C, D \in \mathcal{C}$ are called *opposite*, denoted $C \operatorname{op} D$ if $\delta(C, D) = w_0$.

The notion of opposition extends to all simplices when considering a spherical building as simplicial complex: Let Δ be a spherical building and $\sigma, \tau \in \Delta$. We call σ, τ opposite if there exist chambers (i.e. maximal simplices of Δ) such that $C \operatorname{op} D$, $\sigma \subseteq C$ and $\tau \subseteq D$.

Remark 3.5.13. A spherical Coxeter complex can be visualized as triangulated sphere (and its higher dimensional analogues). In this picture, two chambers are opposite if they are on opposite sides of the sphere. In a general spherical building, given two chambers C, D it suffices to fix one apartment which contains both of them and then consider whether they are opposite in this Coxeter complex. For example in Figure 3.5 opposite to the chamber labeled 1 are the edge $s_1s_2s_1$ and the vertices $s_1s_2\langle s_1\rangle$ and $s_2s_1\langle s_2\rangle$.

The notion of opposition will prove very useful to construct interesting subcomplexes of the spherical buildings. We will consider this in more detail in Section 3.6.

We would also like to be able to define a similar notion for arbitrary buildings. This will not quite be possible, but the idea leads to the definition of twin buildings. Before we consider them, we record a few more facts on chamber systems generated by a group and how their properties can give more insight into the properties of the group.

Definition 3.5.14. Let $(\mathcal{C}, \sim_i, i \in I)$ be a chamber system.

(i) Two galleries $C = C_0 \sim_{i_1} \ldots \sim_{i_n} C_n$, and $D = D_0 \sim_{j_1} \ldots \sim_{j_k} D_k$ are called elementary m-homotopic if there exists a gallery $E = E_0 \sim_{h_1} \ldots \sim_{h_\ell} E_\ell$, indices $0 \leq k_1 < k_2 \leq n$ and a set $J \subseteq I$ with $|J| \leq m$ such that $\{h_1, \ldots, h_\ell\} \subset J, \{i_{k_1+1}, i_{k_1+2}, \ldots i_{k_2}\} \subseteq J$ and such that the gallery D is equal to the following:

$$\begin{array}{ccc} C_0 \sim_{i_1} \ldots \sim_{i_{k_1}} C_{k_1} & & C_{k_2} \sim_{i_{k_2+1}} \ldots \sim_{i_n} C_n. \\ \quad\quad\quad\quad\quad\| & & \| \quad\quad\quad\quad\quad \\ & E_0 \sim_{h_1} \ldots \sim_{h_\ell} E_\ell & \end{array}$$

Intuitively, this means that the two gallery only differ by the middle piece denoted by E and we only change m-many types of equivalence relations. In particular, any two galleries are elementary $|I|$-homotopic.

(ii) We call two galleries m-homotopic if they can be connected by a finite sequence of elementary m-homotopies.

(iii) A chamber system is called m-simply connected if it is connected (i.e. any two chambers can be connected by a gallery) and every closed gallery (i.e. where start and end chamber coincide) is m-homotopic to a gallery of length 0.

Definition 3.5.15 ([Tit86])**.** Let $(G_i)_{i\in I}$ be a family of groups and let $\phi_{ij} : G_i \to G_j$ for $(i, j) \in J \subseteq I \times I$ be a family of homomorphisms.

The direct limit of the system $((G_i)_{i\in I}, (\phi_{ij})_{(i,j)\in J})$ is a group G together with homomorphisms $f_i : G_i \to G$ such that $f_i = f_j \circ \phi_{ij}$ for all $(i, j) \in J$ and which satisfies the following universal property:

Given a group H and homomorphisms $g_i : G_i \to H$ such that $g_i = g_j \circ \phi_{ij}$ for all $(i, j) \in J$ then there exists a unique homomorphism $\alpha : G \to H$ such that $g_i = \alpha \circ f_i$ for all $i \in I$.

The group G can be constructed explicitly by taking the free product of the groups $G_i, i \in I$ and making the identifications $\phi_{ij}(h) = h$ for all $h \in G_i, \forall (i, j) \in J$.

Remark 3.5.16. Let $(I, \leq)$ be a partially ordered set and let $J = \{(i, j) \in I \times I \mid i \leq j\}$. Let $(G_i)_{i\in I}$ be a family of groups with injective homomorphisms $\phi_{ij} : G_i \to G_j$ for all $i \leq j \in I$ such that $\phi_{ij} = \phi_{kj} \circ \phi_{ik}$ whenever $i \leq k \leq j$. Then $((G_i)_{i\in I}, (\phi_{ij})_{(i,j)\in J})$ is called a simple complex of groups, see [BH99, Definition 12.11]. In this case, the direct limit is called the fundamental group of the complex of groups.

Definition 3.5.17. Let $G_i, i \in I$ be a family of subgroups of a group G. We call G the *free product of the G_i amalgamated along their intersections* if G is the direct limit of the system $(\{G_i, G_i \cap G_j \mid i, j \in I\}, \{\phi_{ij} : G_i \cap G_j \hookrightarrow G_i \mid i, j \in I\})$ where ϕ_{ij} are the natural inclusions.

The following result relates the notion of simple-connectedness of certain chamber systems and direct limits. It was first stated by J. Tits in [Tit86], we present it in the formulation of [Sch95, Proposition 6.5.2].

Proposition 3.5.18 ([Sch95, Proposition 6.5.2]). Let G be a group, B and $P_i, i \in I$, subgroups of G such that $B \subset P_i$ for all $i \in I$. Assume that G is generated by the P_i. Set $P_J := \langle P_j : j \in J \rangle$ for all nonempty subsets $J \subseteq I$, and $P_\emptyset := B$. The following are equivalent, for a natural number $m \leqslant |I|$:

(i) the chamber system $\mathcal{C}\left(G, B, P_i, i \in I\right)$ is m-simply-connected;

(ii) G is the direct limit of the $P_J, |J| \leqslant m$ with respect to the natural inclusions.

Remark 3.5.19. In the case when $m = 2$ and $|I| = 3$ the result in Proposition 2.4.7 together with Proposition 3.5.18 show that the chamber system $\mathcal{C}(G, \cap_{i\in I} H_i, H_i, i \in I)$ is 2-simply connected if and only if $\mathcal{CC}(G; (\langle H_j \mid i \in I \setminus \{j\}\rangle)_{i\in I})$ is 1-connected, i.e. the zeroth and first homotopy group of the geometric realization of the coset complex are trivial. The requirement $|I| = 3$ is necessary here to ensure that $\{\{i\}, \{i, j\} \mid i, j \in I\} = \{I \setminus \{i\}, I \setminus \{i, j\} \mid i, j \in I\}$.

For $m \neq 2$, the notions are, to the best of our knowledge, not related. Note that if a chamber system is m-simply connected then it is also $m + 1$-simply connected. On the other hand, if a simplicial complex is n-connected then it is also $n - 1$-connected.

Twin buildings The study of twin buildings was motived by Jacques Tits study of Kac–Moody groups and they were first introduced by Ronan and Tits in the 1980s. We follow [AB08] to give a definition using the chamber system viewpoint of buildings. The central idea is that a twin building consists of a pair $(\mathcal{C}^+, \mathcal{C}^-)$ of two buildings of the same type together with an opposition relation between the chambers of $\mathcal{C}^+$ and those of $\mathcal{C}^-$, with properties similar to those of the opposition relation on the chambers of a spherical building.

Definition 3.5.20. A twin building of type (W, S) is a triple $(\mathcal{C}^+, \mathcal{C}^-, \delta^*)$ consisting of two buildings $(\mathcal{C}^+, \delta^+)$ and $(\mathcal{C}^-, \delta^-)$ of type (W, S) together with a codistance function

$$\delta^* : \left(\mathcal{C}^+ \times \mathcal{C}^-\right) \cup \left(\mathcal{C}^- \times \mathcal{C}^+\right) \to W$$

satisfying the following conditions for each $\epsilon \in \{+, -\}$, any $C \in \mathcal{C}^\epsilon$, and any $D \in \mathcal{C}^{-\epsilon}$, with $w := \delta^*(C, D)$:

(i) $\delta^*(C, D) = \delta^*(D, C)^{-1}$.

(ii) If $C' \in \mathcal{C}^\epsilon$ satisfies $\delta^\epsilon\left(C', C\right) = s$ with $s \in S$ and length$(sw) <$ length(w), then $\delta^*\left(C', D\right) = sw$.

(iii) For any $s \in S$, there exists a chamber $C' \in \mathcal{C}^\epsilon$ with $\delta^\epsilon\left(C', C\right) = s$ and $\delta^*\left(C', D\right) = sw$.

Definition 3.5.21. Given a twin building $(\mathcal{C}^+, \mathcal{C}^-)$ and chambers $C \in \mathcal{C}^+, D \in \mathcal{C}^-$ we say that C is *opposite* D if $\delta^*(C, D) = 1$.

We extend this definition again to lower dimensional simplices in the simplicial complex viewpoint of buildings and say that two simplices are opposite if they are contained in opposite chambers.

Remark 3.5.22. Some authors define that two lower dimensional simplices are opposite only if they are of the same type and contained in opposite chambers. We will not require this here, since we want the collection of simplices opposite one fixed chamber to form a simplicial complex.

Remark 3.5.23. Intuitively, two chambers are further away in the twin building if their codistance is smaller with respect to the word length in (W, S).

Example 3.5.24. Every spherical building gives rise to a twin building. Let $\mathcal{C}$ be a spherical building with Weyl distance δ. Let $\mathcal{C}^{\pm}$ be two copies of $\mathcal{C}$ and for a chamber $C \in \mathcal{C}$, let $C_{\pm} \in \mathcal{C}^{\pm}$ denote the corresponding copies. Then we define

$$\delta^{+}(C_{+}, D_{+}) = \delta(C, D), \qquad \delta^{-}(C_{-}, D_{-}) = w_0 \delta(C, D) w_0$$

where w_0 is the longest word of the Coxeter group. Note that $(\mathcal{C}^{-}, \delta^{-})$ is called the dual of $(\mathcal{C}, \delta)$. They are both buildings of type (W, S). The map $\phi : \mathcal{C} \to \mathcal{C}^{-} : C \mapsto C_{-}$ is a σ_0-isometry, meaning that $\delta^{-}(\phi(C, D)) = \sigma_0(\delta(C, D))$ with $\sigma_0 : W \to W : w \mapsto w_0 w w_0$. The corresponding simplicial complexes are isomorphic but not type-preservingly isomorphic, see [AB08, Section 5.7.7] for more details.

We get a twin building $(\mathcal{C}^{+}, \mathcal{C}^{-}, \delta^{*})$ by setting

$$\delta^{*}(C_{+}, D_{-}) = \delta(C, D) w_0, \qquad \delta^{*}(D_{-}, C_{+}) = w_0 \delta(D, C).$$

Note that the opposition relation coincides with the one defined in Definition 3.5.12: $\delta^{*}(C_{+}, D_{-}) = 1 \iff \delta(C, D) = w_0^{-1} = w_0 \iff \delta^{*}(D_{-}, C_{+}) = 1$.

RGD-systems We finally explain the relationship between Chevalley groups and spherical buildings and Kac–Moody groups and twin buildings.

This connection occurs since both Chevalley and Kac–Moody groups give rise to a linear RGD system (RGD stands for “root group datum”). We give the definition following [Mar18, Definition B.35] and restrict to the case of linear RGD systems, since they allow for a nice description of the parabolic subgroups, see Lemma 3.5.28. Since we later are only interested in RGD systems coming from Chevalley and Kac–Moody groups, this is sufficient. Note that the “linear” only affects (RGD1).

Definition 3.5.25. Let (W, S) be a Coxeter system where W is the Weyl group associated to a GCM with root system Φ. A linear *RGS system* of type (W, S) is a triple $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ consisting of a group G and subgroups $U_\alpha \leq G$, and $T \leq G$ such that

(RGD0) $U_\alpha \neq \{1\}$ for all $\alpha \in \Phi$. We call U_α a *root subgroup.*

(RGD1) $[U_\alpha, U_\beta] \subseteq \langle U_\gamma \mid \gamma \in]\alpha, \beta[_{\mathbb{N}} \rangle$ for each prenilpotent pair $\alpha, \beta \in \Phi$.

(RGD2) For every $s \in S$ there is a function $m : U_{\alpha_s} \setminus \{1\} \to G$ such that for all $u \in U_{\alpha_s} \setminus \{1\}$ and $\alpha \in \Phi$,

$$m(u) \in U_{-\alpha_s} u U_{-\alpha_s} \quad \text{and} \quad m(u) U_\alpha m(u)^{-1} = U_{s\alpha}.$$

Moreover, $m(u)^{-1} m(v) \in T$ for all $u, v \in U_{\alpha_s} \setminus \{1\}$.

(RGD3) For every $s \in S$ we have $U_{-\alpha_s} \not\subseteq U^+ = \langle U_\gamma \mid \gamma \in \Phi^+ \rangle$.

(RGD4) $G = T\langle U_\alpha \mid \alpha \in \Phi \rangle$.

(RGD5) T normalizes each root subgroup U_α.

Given an RGD-system $(G, (U_\alpha)_{\alpha\in\Phi}, T)$, we define the following groups.

- $U^+ := \langle U_\alpha \mid \alpha \in \Phi^+ \rangle, U^- := \langle U_\alpha \mid \alpha \in \Phi^- \rangle$,
- positive and negative Borel subgroup $B^+ := \langle T, U^+ \rangle$, $B^- := \langle T, U^- \rangle$,
- $N := \langle T, m(u); u \in U_{\alpha_s} \setminus \{1\}, s \in S \rangle$.

Proposition 3.5.26 ([AB08, Theorem 8.80])**.** Given an RGD-system $(G, (U_\alpha)_{\alpha\in\Phi}, T)$, we have that $(B^+, N), (B^-, N)$ are two BN-pairs. Furthermore, they form what is called a twin Tits system, and give rise to a twin building $\mathcal{C}^+, \mathcal{C}^-$ with

(i) $\mathcal{C}^\pm = G/B^\pm$ with Weyl distance given by the individual BN-pairs,

(ii) for $\epsilon \in \{+, -\}, g, h \in G, w \in W$ we have

$$\delta^*(gB^\epsilon, hB^{-\epsilon}) = w \iff g^{-1}h \in B^\epsilon w B^{-\epsilon}.$$

Proposition 3.5.27 (See e.g. [Mar18, Theorem 7.69])**.** Let A be a GCM, and K a field. Let $G_A(K)$ be the associated minimal Kac–Moody group with real roots Φ, associated root subgroups $U_\alpha, \alpha \in \Phi$ and torus T_K. Then $(G_A(K), (U_\alpha)_{\alpha\in\Phi}, T_K)$ is a linear RGD-system.

Lemma 3.5.28. Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be a linear RGD-system with set of simple roots $\Pi = \{\alpha_i \mid i \in I\}$. For $J \subseteq I$ set $\Phi_J = \Phi \cap \oplus_{i\in J}\mathbb{Z}\alpha_i$. Then the parabolic subgroups of the associated twin BN-pair, $P_J^\pm = B^\pm W_J B^\pm$ can be described as

$$P_J^\pm = \langle B^\pm, U_{\mp\alpha} \mid \alpha \in \Phi_J \rangle.$$

Proof. This follows for example from [Mar18, Theorem B.39]. □

Remark 3.5.29. In particular, since every Chevalley group is a Kac–Moody group of finite type, we get a linear RGD-system for each Chevalley group as well. The associated buildings will be spherical, thus one usually considers only one copy. We will consider it a bit more carefully to show that the twinning that we get from the RGD-system coincides with the natural one for two copies of spherical buildings as in Example 3.5.24. Let $G = G_A(K)$ be a Chevalley group with twin BN-pair (B^+, B^-, N) as above. The map $f : G/B^+ \to G/B^- : gB^+ \mapsto gw_0B^-$ defines a σ_0-isometry between the chamber systems $\mathcal{C}^+$ and $\mathcal{C}^-$ since

$$\begin{aligned} \delta^+(gB^+, hB^+) = w &\iff g^{-1}h \in B^+wB^+ \\ &\iff (gw_0)^{-1}hw_0 \in w_0B^+wB^+w_0 = B^-w_0ww_0B^- \end{aligned}$$

and hence $\delta^-(gw_0B^-, hw_0B^-) = w_0\delta^+(gB^+, hB^+)w_0$. In particular, the associated simplicial complexes are identical. Furthermore, we have

$$\delta^*(gB^+, hw_0B^-) = w \iff g^{-1}hw_0 \in B^+wB^- \iff g^{-1}h \in B^+ww_0B^+$$

and thus $\delta^*(gB^+, hw_0B^-) = \delta(gB^+, hB^+)w_0$ as in Example 3.5.24 and similarly

$$\delta^*(hw_0B^-, gB^+) = w_0\delta(gB^+, hB^+).$$

3.6 Opposite complex

In this section, we take a closer look at the opposition relation and at the complex of simplices opposite one fixed chamber. We describe how this complex can be seen as a coset complex over the group U^+ generated by all positive root subgroups. We use this geometric viewpoint on U^+ to deduce some of its properties.

For this section, we fix a 2-spherical GCM $A = (A_{ij})_{i,j\in I}$ with real roots Φ, simple roots $\Pi = \{\alpha_i \mid i \in I\}$ and with Coxeter system (W, S), $S = \{s_i \mid i \in I\}$, and a field K. Furthermore, let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be a linear RGD-system of type (W, S). Since $(B^+, N), (B^-, N)$ are two BN-pairs they give rise to the standard parabolic subgroups $P_J^\pm = B^\pm W_J B^\pm = \langle B^\pm, U_{\mp\alpha} \mid \alpha \in \Phi_J\rangle$ for all $J \subseteq I$. As chamber systems, the corresponding buildings can be described as

$$\mathcal{C}^+ = \mathcal{C}(G, B^+, P^+_{\{i\}}; i \in I), \qquad \mathcal{C}^- = \mathcal{C}(G, B^-, P^-_{\{i\}}; i \in I).$$

They also can be described using the terminology of coset complexes (compare to Remark 3.5.7)

$$\Delta^+ = \mathcal{CC}(G; (P^+_{I\setminus\{i\}})_{i\in I}), \qquad \Delta^- = \mathcal{CC}(G; (P^-_{I\setminus\{i\}})_{i\in I}).$$

Definition 3.6.1. (i) We call the chamber $1B^+$ the positive fundamental chamber.

(ii) We denote by $\mathcal{C}^-_{\text{op}} \subseteq \mathcal{C}^-$ the set of chambers opposite $1B^+$, i.e.

$$\mathcal{C}^-_{\text{op}} = \{C \in \mathcal{C}^- \mid C \operatorname{op} 1B^+\}.$$

(iii) Analogously, we denote by $\Delta^-_{\text{op}} \subseteq \Delta^-$ the complex opposite $1B^+$, i.e.

$$\Delta^-_{\text{op}} = \{\tau \in \Delta^- \mid \tau \operatorname{op} \{1P^+_{I\setminus\{i\}} \mid i \in I\} \leftrightarrow 1B^+\}.$$

For short, we will also call Δ^-_{op} the *opposite complex.*

Remark 3.6.2. If $\mathcal{C}$ is a spherical building, we can define $\mathcal{C}_{\text{op}}(C_0) = \{D \in \mathcal{C} \mid D \operatorname{op} C_0\}$ for any chamber $C_0 \in \mathcal{C}$ and similarly for the simplicial complex viewpoint $\Delta_{\text{op}}(C_0)$. If the RGD-system that we started with is of spherical type, then the σ_0-isometry from Remark 3.5.29 restricts to a σ_0-isometry from $\mathcal{C}_{\text{op}}(1B^+)$ to $\mathcal{C}^-_{\text{op}}$ and $\Delta_{\text{op}}(C_0)$ is isomorphic to Δ^-_{op}. Hence we will in the remainder of this section only consider the opposite complex for a twin building but all the results will apply also to the opposite complex in a single spherical building associated to a Chevalley group.

Lemma 3.6.3 ([AB08, Corollary 8.32])**.** The group U^+ acts sharply transitively on the set $\mathcal{C}^-_{\mathrm{op}}$ of chambers opposite the positive fundamental chamber.

Let $\mathcal{U}_J := \langle U_\alpha \mid \alpha \in \Phi^+ \cap \oplus_{j\in J}\mathbb{Z}\alpha_j\rangle \leq G$ and $U_J = \langle U_{\alpha_j} \mid j \in J\rangle \leq G$. Clearly $U_J \subseteq \mathcal{U}_J$ but the two groups are not necessary equal. In case $J = \{j\}$ is a singleton, the two sets coincide. For RGD-systems from Kac–Moody groups we will investigate this further in Proposition 3.6.8. The following lemma is well known to experts, but we could not find a reference for the statement, hence we give a short proof.

Lemma 3.6.4. Let $\emptyset \neq J \subsetneq I$ be a proper subset of I. Then

$$U^+ \cap P_J^- = \mathcal{U}_J.$$

Proof. Since $P_J^- = \langle B^-, U_\alpha \mid \alpha \in \Phi_J\rangle$ we have $\mathcal{U}_J \subseteq U^+ \cap P_J^-$.

To see the other inclusion, let $C_- = 1B^-$ be the negative fundamental chamber and consider its J-residue in $\mathcal{C}^-$.

$$R_J(C_-) = \{hB^- \mid h \in \langle P^-_{\{j\}} \mid j \in J\rangle\} = \{hB^- \mid h \in P_J^-\}.$$

Its stabilizer in $G = G_A(K)$ is $\mathrm{Stab}_G(R_J(C_-)) = P_J^-$.

It follows from [AB08, Corollary 5.30] that $R_J(C_-) = \mathcal{C}(P_J^-, B^-, P_j, j \in J) =: \mathcal{C}_J^-$ is a building in its own rights, of type $A_J := (A_{ij})_{i,j\in J}$. Similarly, we get the building $\mathcal{C}_J^+ := \mathcal{C}(P_J^+, B^+, P_j^+, j \in J)$. This pair gives again rise to a twin building such that $\mathcal{C}^-_{J,\mathrm{op}} \cong R_J(C_-) \cap \mathcal{C}^-_{\mathrm{op}}$.

Further note that the group $\mathcal{U}_J$ is the U^+ for the smaller twin building of type A_J.

Now, let $g \in U^+ \cap P_J^-$. Then g stabilizes $R_J(C_-)$ by the above results and thus acts on $\mathcal{C}_J^-$. Furthermore, gC_- is still opposite c_+ since $g \in U^+$. Hence $gC_- \in R_J(C_-) \cap \mathcal{C}^-_{\mathrm{op}}$. By Lemma 3.6.3 and the above considerations, $\mathcal{U}_J$ acts transitively on $R_J(C_-) \cap \mathcal{C}^-_{\mathrm{op}}$. Thus we can find $u \in \mathcal{U}_J$ such that $ugC_- = C_-$. Hence $ug \in U^+ \cap \mathrm{Stab}_G(C_-) = U^+ \cap B^- = \{1\}$, the last equality follows from [AB08, Section 8.7]. Therefore we conclude $g \in \mathcal{U}_J$ as desired. □

The result leads to the following observation.

Lemma 3.6.5. For all $J, L \subseteq I$ we have

$$\mathcal{U}_J \cap \mathcal{U}_L = \mathcal{U}_{J\cap L}.$$

Proof. From Lemma 3.5.3, we have for any $J, L \subseteq I$ that $P_J^- \cap P_L^- = P^-_{J\cap L}$. Combining this with Lemma 3.6.4, we get

$$\mathcal{U}_J \cap \mathcal{U}_L = (U^+ \cap P_J^-) \cap (U^+ \cap P_L^-) = U^+ \cap (P_J^- \cap P_L^-) = U^+ \cap P^-_{J\cap L} = \mathcal{U}_{J\cap L}.$$

□

The following proposition is well-known to expert, but not mentioned explicitly in our notation in the literature. Since it is essential for the following results we give a proof. The analogous result for spherical buildings was observed in [Abr96, Page 67]

Proposition 3.6.6. Let $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ be a linear RGD-system. Then

$$\mathcal{C}_{\mathrm{op}}^- = \mathcal{C}(U^+B^-, B^-, P_{\{i\}}, i \in I) \cong \mathcal{C}(U^+, \{1\}, U_{\alpha_i}; i \in I)$$

as chamber system. As simplicial complex, we have

$$\Delta_{\mathrm{op}}^- \cong \mathcal{CC}(U^+; (\mathcal{U}_{I\setminus\{i\}})_{i\in I}).$$

Remark 3.6.7. If $U_J \neq \mathcal{U}_J$ for some $J \subset I$, then the correspondence between chamber systems and coset complexes still works for $\mathcal{C}(U^+B^-, B^-, P_{\{i\}}, i \in I)$ and $\mathcal{CC}(U^+; (\mathcal{U}_{I\setminus\{i\}})_{i\in I})$ but fails when considering $\mathcal{C}(U^+, \{1\}, U_{\alpha_i}; i \in I)$. If $\mathcal{U}_J = U_J$ for all $J \subseteq I$ then the correspondence also works in that case.

Proof. By Lemma 3.6.3 U^+ acts sharply transitively on $\mathcal{C}_{\mathrm{op}}^-$. Clearly, we have $1B^- \in \mathcal{C}_{\mathrm{op}}^-$. Hence we get that as sets $\mathcal{C}_{\mathrm{op}}^- = \{uB^- \mid u \in U^+\} \cong U^+$. Now two chambers uB^-, vB^- are i-adjacent (i.e. $uB^- \sim_i vB^-$) if and only if $u^{-1}v \in P_{\{i\}}^-$ and since $u^{-1}v \in U^+$ this is equivalent to $u^{-1}v \in U^+ \cap P_{\{i\}}^- = U_{\alpha_i}$. Thus, as chamber systems we have $\mathcal{C}_{\mathrm{op}}^- \cong \mathcal{C}(U^+, \{1\}, U_{\alpha_i}; i \in I)$.

For the simplicial complex approach, we use Remark 2.4.4. We know that U^+ acts transitively on maximal simplices. We fix $1B^- \leftrightarrow \{1P_{I\setminus\{i\}}^- \mid i \in I\}$ and consider the stabilizers of the vertices. These are $\mathrm{Stab}_{U^+}(1P_{I\setminus\{i\}}) = P_{I\setminus\{i\}}^- \cap U^+ = \mathcal{U}_J$. Thus we get the desired description as coset complex.

If $\mathcal{U}_J = U_J$ then the $(U_{I\setminus\{i\}})_{i\in I}$ satisfy the assumptions of Remark 3.5.7 and the correspondence described there connects $\mathcal{C}(U^+, \{1\}, U_{\alpha_i}; i \in I)$ and $\mathcal{CC}(U^+; (U_{I\setminus\{i\}})_{i\in I})$. $\square$

This geometric viewpoint on the group U^+ can be used to prove some useful properties. The following proposition is well-known to experts, but the proof is scattered over different sources. We gather here all the pieces. Note that part (b) is a direct analogue to [DM07, Corollary 1.2]. In this formulation, it was first mentioned in [Ers08, Theorem 6.2].

Proposition 3.6.8. Let K be an arbitrary field, A a GCM, $G_A(K)$ the corresponding Kac–Moody group with RGD-system $(G_A(K), (U_\alpha)_{\alpha\in\Phi}, T_K)$ and $U^+ = \langle U_\alpha \mid \alpha \in \Phi^+\rangle$ as above.

(a) If A is 2-spherical and $|K| \geq 4$, then $U^+ = \langle U_{\alpha_i} \mid i \in I\rangle$ i.e. it is generated by the simple root subgroups.

(b) If A is 3-spherical and $|K| \geq 5$, then U^+ is the direct limit of the groups $U_J = \langle U_{\alpha_j} \mid j \in J\rangle$ with $1 \leq |J| \leq 2$ with respect to the natural inclusions $U_{\{i\}} \hookrightarrow U_{\{i,j\}}$.

Sketch of the proof. For part (a), note that $\mathcal{C}_{\mathrm{op}}^-$ is gallery-connected if and only if $U^+ = \langle U_{\alpha_i} \mid i \in I\rangle$. The fact that $\mathcal{C}_{\mathrm{op}}^-$ is gallery-connected under the assumption that $|K| \geq 4$ and that A is 2-spherical can be found in [AM97].

For part (b), we want to apply Proposition 3.5.18 for $m = 2$ to $\mathcal{C}_{\mathrm{op}}^-$. Thus, we need to show that $\mathcal{C}_{\mathrm{op}}^-$ is simply 2-connected. [DM07, Theorem 1.1] reduces the problem to the following

- for each rank 3 residue R containing the positive fundamental chamber C, the set of chambers opposite of C in R is simply 2-connected;
- for each rank 2 residue R containing the positive fundamental chamber C, the set of chambers opposite of C in R is connected.

Since we assume that A is 3-spherical, each rank 3 and rank 2 residue will itself be contained in a spherical building.

The required results for the rank 3 and rank 2 residues can be found in [HNV16], generalizing results of [Abr96]. □

Remark 3.6.9. In particular, if A is 2-spherical and $|K|\geq 4$ then $U_J = \mathcal{U}_J$ for all $J \subseteq I$.

Lastly, we discuss the residual finiteness of U^+.

Theorem 3.6.10. Let A be a 2-spherical GCM and let K be a finite field of characteristic p. Then the group $U^+ = U_A^+(K)$ is residually-p.

Proof. This is a consequence of [RR06, 1.C Lemma 1]. For the readers convenience, we provide an argument for the residually finiteness of U^+.

We have that U^+ is contained in the stabilizer of the positive fundamental chamber in the positive building associated to the Kac–Moody group. Thus, the action of U^+ stabilizes balls of radius $r \in \mathbb{N}$ around the fundamental chamber in $\mathcal{C}^+$. The point-wise fixers of these balls are then finite index normal subgroups of U^+. Since U^+ acts faithfully on the building, the conclusion follows. □

3.7 Geometric construction of buildings of type A_n, C_n, D_n

In the following section, we describe how buildings of type A_n, C_n and D_n can be constructed as flag complexes of certain sets equipped with an incidence relation. We furthermore describe the opposite complexes in this set-up. We follow [Abr96]. The general framework is as follows.

Definition 3.7.1. Let X be a set and let $I \subseteq X \times X$ be a reflexive and symmetric relation. For $a, b \in X$ we write aIb if and only if $(a, b) \in I$ and call a and b incident. The structure gives rise to the following simplicial complex, called the flag complex of X:

$$\operatorname{Flag} X = \{\sigma \subseteq X \mid \forall a, b \in \sigma : aIb\}.$$

Note that a flag complex X is a clique complex, i.e. a set of vertices $\{v_0, \ldots, v_k\}$ forms a simplex in X if and only if they are pairwise connected by an edge, i.e $\{v_i, v_j\} \in X$ for all $0 \leq i < j \leq k$.

Buildings of type A_n Let $n \in \mathbb{N}, n \geq 1$. Any building of type A_n can be described in the following way, see e.g. [Tit74]. Let K be a field or skew field, V an $(n+2)$-dimensional vector space over K. Set $X := X(V) := \{0 < U < V \mid U \text{ is a } K\text{-subspace of } V\}$. Two subspaces $U, W \in X$ are called incident if $U \subseteq W$ or $W \subseteq U$. Then $\Delta = \operatorname{Flag} X$ is a building of type A_n.

To describe the opposite complex in this context, we need the following notation.

Definition 3.7.2 ([Abr96, Definition 9]). (i) Two subspaces U, W of V are called transversal (in V) if $U \cap W = 0$ or $U + W = V$. If this is the case, we write $U \pitchfork_V W$ or simply $U \pitchfork W$.

(ii) Let $\mathcal{E}$ be a set of subspaces of V. Then we write $U \pitchfork \mathcal{E}$ if $U \pitchfork E$ for all $E \in \mathcal{E}$. We set

$$X_{\mathcal{E}}(V) := \{U \in X \mid U \pitchfork \mathcal{E}\} \qquad T_{\mathcal{E}}(V) = \operatorname{Flag} X_{\mathcal{E}}(V).$$

We get the following result.

Proposition 3.7.3 ([Abr96, Corollary 12]). For any simplex $\sigma = \{E_1 < \cdots < E_r\} \in \Delta$ set $\mathcal{E}(\sigma) = \{E_i \mid 1 \leq i \leq r\}$. Then

$$\Delta_{\mathrm{op}}(\sigma) = T_{\mathcal{E}(\sigma)}(V).$$

Definition 3.7.4. We denote by $\mathbf{C}_A$ the class of all simplicial complexes $T_{\mathcal{E}}(V)$ where $n := \dim(T_{\mathcal{E}}(V)) \geq 0$, V an $(n+2)$-dimensional vector space over a (skew) field K, $\mathcal{E}$ is a finite set of subspaces of V such that if we set $e_j = |\{E \in \mathcal{E} \mid \dim E = j\}|$ we have $\sum_{j=1}^{n+1} \binom{n}{j-1} e_j \leq |K|$.

Hermitian and pseudo-quadratic forms In this part, we recall facts about hermitian and pseudo-quadratic forms that are needed to describe the buildings of type C_n and D_n. We will state all the necessary technicalities, but not explain much on the intuition, since it is not necessary for our applications. We follow [Abr96, Chapter 5], who in turn mostly follows [Tit74].

Let K be a skew field, $\sigma : K \to K, a \mapsto a^\sigma$ be an involution, i.e. an anti-automorphism of K such that $\sigma^2 = \mathrm{id}_K$. Let $\epsilon \in \{1, -1\} \subset K$. If $\sigma \neq \mathrm{id}_K$ then we require $\epsilon = -1$. Let V be a right K-vector space of dimension $m \in \mathbb{N} \cup \{\infty\}$. Let $K_{\sigma,\epsilon} := \{\alpha - \alpha^\sigma \varepsilon \mid \alpha \in K\}$ and $K^{\sigma,\epsilon} := \{\alpha \in K \mid \alpha + \alpha^\sigma \varepsilon = 0\}$. Let Λ be a form parameter relative to (σ, ε), i.e. Λ is a subgroup of $(K, +)$ satisfying $K_{\sigma,\varepsilon} \subseteq \Lambda \subseteq K^{\sigma,\varepsilon}$ and $\alpha^\sigma \Lambda \alpha \subseteq \Lambda$ for all $\alpha \in K$.

Let $f : V \times V \to K$ be a (σ, ϵ)-hermitian form, i.e. f is biadditive and for all $x, y \in V, a, b \in K$ we have $f(xa, yb) = a^\sigma f(x, y) b$ and $f(y, x) = f(x, y)^\sigma \epsilon$. Let $Q : V \to K/\Lambda$ be a (σ, ϵ)-quadratic form with associated (σ, ϵ)-hermitian form f, i.e. $Q(xa) = a^\sigma Q(x) a + \Lambda$ and $Q(x + y) - Q(x) - Q(y) = f(x, y) + \Lambda$ for all $x, y \in V, a \in K$. If $\Lambda = K$ we require that f is alternating. If $\Lambda \neq K$ then f is uniquely defined by Q.

For $M \subseteq V$ we write $M^\perp := \{x \in V \mid f(x, M) = 0\}$. A subspace $U < V$ is called

- non-degenerate if $U \cap U^\perp = 0$,

- totally degenerate if $U \subseteq U^{\perp}$,
- anisotropic if $0 \notin Q(U \setminus \{0\})$,
- isotropic if $0 \in Q(U \setminus \{0\})$,
- totally isotropic if $U \subseteq U^{\perp}$ and $Q(U) = 0$.

We require that there is at least one finite-dimensional, maximal, totally isotropic subspace. In that case, all maximal, totally isotropic subspaces have the same dimension. We denote this dimension by n and call it the Witt index of (V, Q, f). We require $0 < n < \infty$. Furthermore, we require that $V^{\perp} = 0$. We say that the triple (V, Q, f) is a pseudo-quadratic space if it is of the form described above.

If it additionally satisfies $(m, \Lambda) \neq (2n, 0)$ then we say (V, Q, f) is a thick pseudo-quadratic space.

Note that if (V, Q, f) is a thick pseudo-quadratic space and K is a finite field, then $\dim V \leq 2n + 1$. In general, if one only considers finite fields, especially of characteristic different from 2, the set-up can be substantially simplified, see [AB08, Remark 9.3, 9.4].

Buildings of type C_n Let $n \in \mathbb{N}, n \geq 1$ and (V, Q, f) be a thick pseudo-quadratic space with Witt index n. Set $X := X(V) = \{0 < U < V \mid U \text{ is totally isotropic}\}$ with incidence relation given by containment as in the A_n case and $\Delta = \operatorname{Flag} X$. Then Δ is a thick building of type C_n and every classical C_n building can be obtained in this way (here classical means that the links of type A_2 correspond to Desarguesian planes, which is always the case for $n \geq 4$), see [Tit74, Theorem 8.22]. We start with the following technical definition, which is used to describe when a building is "thick enough".

Definition 3.7.5. Set $\mathcal{U} := \{0 \leq U \leq V \mid \dim U < \infty\}$, $\mathcal{U}^{\perp} := \{U^{\perp} \mid u \in \mathcal{U}\}$ and $\mathcal{W} := \mathcal{U} \cup \mathcal{U}^{\perp}$.

Let $\mathcal{E}$ be a finite subset of $\mathcal{W}$ such that $\mathcal{E}^{\perp} = \mathcal{E}$. Assume $K = \mathbb{F}_q$ is a finite field. Set $\mathcal{E}_j := \{E \in \mathcal{E} \mid \dim E = j\}, e_j := |\mathcal{E}_j|$ and $e_h^{(s)} := \sum_{j=0}^{2s} \binom{2s}{j} e_{h+j}$ for $h \in \mathbb{N}, s \in \mathbb{N}_0$ and $h + 2s < m$. We define

- $N(\mathcal{E}) := e_1^{(n-1)}$ if f is alternating ($\Rightarrow m = 2n$) and $Q = 0$,
- $N(\mathcal{E}) := \left(e_1^{(n-1)}\right)^2$ if $m = 2n$ and $\sigma \neq \mathrm{id}$,
- $N(\mathcal{E}) := 2e_2^{(n-1)}$ if $m = 2n + 1$,
- $N(\mathcal{E}) := \max\left\{e_2^{(n-1)} + e_3^{(n-1)} + 1, 2e_3^{(n-1)}\right\}$ if $m = 2n + 2$.

Definition 3.7.6. For a set $\mathcal{E}$ of subspaces of V we again define

$$X_{\mathcal{E}}(V) := \{U \in X \mid U \pitchfork \mathcal{E}\}, \qquad T_{\mathcal{E}}(V) = \operatorname{Flag} X_{\mathcal{E}}(V).$$

This is very similar to Definition 3.7.2, but here the set X is different since it only contains proper totally isotropic subspaces and not all proper subspaces.

Proposition 3.7.7 ([Abr96, Corollary 15])**.** For any simplex $\tau = \{E_1 < \cdots < E_r\} \in \Delta$ set $\mathcal{E}(\tau) = \{E_i, E_i^\perp \mid 1 \leq i \leq r\}$. Then $\Delta_{\text{op}}(\tau) = T_{\mathcal{E}(\tau)}(V)$.

Definition 3.7.8. With $\mathbf{C}_C$ we denote the class of simplicial complexes $T_{\mathcal{E}}(V)$ where $n = \dim(T_{\mathcal{E}}(V)) + 1 \geq 1$, K a (skew) field, (V, Q, f) is a thick pseudo-quadratic space with Witt index n, $\mathcal{E}$ a finite subset of $\mathcal{W}$ (see Definition 3.7.5) such that $\mathcal{E}^\perp = \mathcal{E}$. If K is a finite field, we further require that $|K| \geq N(\mathcal{E})$.

Buildings of type D_n For this section let (V, Q, f) be a pseudo-quadratic space but in the specific case where K is a field, $\sigma = \mathrm{id}_K, \epsilon = 1, \Lambda = 0$, V a K-vector space of dimension $m = 2n \geq 4$, Q an ordinary quadratic form and f the corresponding symmetric bilinear form. We further assume that V is non-degenerate and of Witt index n.

We set as before $X = X(V) = \{0 < U < V \mid U \text{ is totally isotropic}\}$. Then $\Delta = \operatorname{Flag} X$ is a weak C_n building, in particular it is not thick, since each totally isotropic subspace of dimension $n-1$ is contained in exactly two totally isotropic subspaces of dimension n. Furthermore, the space $Y = \{U \in X \mid \dim U = n\}$ is partitioned into two sets $Y = Y_1 \sqcup Y_2$ such that for $U_1, U_2 \in Y_i$ we have $n - \dim(U_1 \cap U_2) \in 2\mathbb{Z}$ for $i = 1, 2$ and for $U_1 \in Y_1, U_2 \in Y_2$ we have $n - \dim(U_1 \cap U_2) \in 2\mathbb{Z} + 1$. In particular, if for $U_1, U_2 \in Y$ we have $\dim(U_1 \cap U_2) = n - 1$ then U_1 and U_2 are in different sets of the partition.

To obtain a thick building of type D_n we set $\tilde{X} = \tilde{X}(V) = \{U \in X \mid \dim U \neq n - 1\}$ and we say that $U, W \in \tilde{X}$ are incident if and only if

$$U \subseteq W \text{ or } W \subseteq U \text{ or } \dim(W \cap U) = n - 1.$$

Then $\tilde{\Delta} = \operatorname{Flag} \tilde{X}$ is the desired building. We will write $\operatorname{Orifl}(\tilde{X})$ for $\operatorname{Flag} \tilde{X}$ to stress that we consider $\tilde{X}$ not with the usual incidence relation given by inclusion, but with this new one. Note that $\operatorname{Orifl} \tilde{X}$ is called the oriflamme complex of $\tilde{X}$.

For $n \geq 4$ every building of type D_n is of the form $\tilde{\Delta}$ for some field K [Tit74, Proposition 8.4.3]. We also consider the construction for $n = 2, 3$ in which case we get certain (but not all) buildings of type $D_2 = A_1 \times A_1$ and $D_3 = A_3$, which were already covered earlier.

To describe the opposite complexes, we need to introduce some further notation.

Definition 3.7.9. For arbitrary subspaces U, W of V, we define

$$U \tilde{\pitchfork}_V W \iff U \pitchfork_V W \text{ or } (U, W \in \tilde{X}, U = U^\perp, W = W^\perp \text{ and } \dim(U \cap W) = 1).$$

For a set $\mathcal{E}$ of subsets of V we define (different from Definition 3.7.2)

$$\begin{aligned} X_{\mathcal{E}}(V) &= \{U \in X \mid U \tilde{\pitchfork} \mathcal{E}\}, & T_{\mathcal{E}}(V) &= \operatorname{Flag} X_{\mathcal{E}}(V) \\ \tilde{X}_{\mathcal{E}}(V) &= \{U \in \tilde{X} \mid U \tilde{\pitchfork} \mathcal{E}\}, & \tilde{T}_{\mathcal{E}}(V) &= \operatorname{Orifl} \tilde{X}_{\mathcal{E}}(V) \end{aligned}$$

Proposition 3.7.10 ([Abr96, Corollary 17])**.** Let $\tau = \{E_1 < \cdots < E_r\} \in \tilde{\Delta}$ and set $\mathcal{E}(\tau) = \{E_i, E_i^\perp \mid 1 \leq i \leq r\}$ then $\tilde{\Delta}_{\text{op}}(\tau) = \tilde{T}_{\mathcal{E}(\tau)}(V)$.

Chapter 4

Kac–Moody–Steinberg groups and complexes

We are now ready to define the main objects of our work, the KMS groups and complexes. A significant part of this chapter appeared in [GV25].

4.1 Kac–Moody–Steinberg groups

For this chapter we fix a 2-spherical GCM $A = (A_{ij})_{i,j\in I}$ with real roots Φ and simple roots $\Pi = \{\alpha_i \mid i \in I\}$. Furthermore, we fix a field K of size at least 4. We denote by $G_A(K)$ the corresponding minimal Kac–Moody group with root subgroups $U_\alpha, \alpha \in \Phi$.

Definition 4.1.1. Let $Q_A = \{J \subseteq I \mid A_J = (A_{ij})_{i,j\in J} \text{ is spherical}\}$ be the set of all spherical subsets of I. As before, we write $U_J = \langle U_{\alpha_i} \mid i \in J\rangle \leq G_A(K)$ for all $J \in Q_A$. For $J \subseteq L \in Q_A$ we have an injective homomorphism $\phi_{J,L} : U_J \to U_L$ given by the inclusion $U_J \subseteq U_L$.

The Kac–Moody–Steinberg (KMS) group of type A over K is the direct limit of the system $((U_J)_{J\in Q_A}, (\phi_{J,L})_{J\subseteq L\in I})$ (see Definition 3.5.15). We denote it by $\mathcal{U}_A(K)$.

The first appearance of a Kac–Moody–Steinberg group was in [EJ10] as a Golod-Shafarevich group with property (T). The rank 3 case was further investigated in [Cap+22]. KMS groups over $\mathbb{Q}$ and in particular their relationship to the maximal unipotent subgroup U^+ of the corresponding Kac–Moody group were studied in [CM26]. Note that there is a dichotomy between the 3-spherical case and the 2-spherical but not 3-spherical case. In the first case, the KMS group is isomorphic to U^+ which is not the case otherwise.

Example 4.1.2. We consider the KMS group of type $\mathrm{HB}_2^{(2)}$ that will give an interesting construction of KMS complexes later. The corresponding Dynkin diagram can be seen in Figure 4.1. The corresponding system of groups and morphisms is displayed by the diagram in Figure 4.2 which is an instance of what is called a triangle of groups.

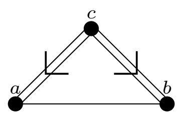


Figure 4.1: Dynkin diagram of type $\mathrm{HB}_2^{(2)}$.

$$\begin{tikzcd}
U_{ab} & & U_b \arrow[ll, "\phi_{ba}"'] \arrow[rr, "\phi_{bc}"] & & U_{bc} \\
& & \{1\} \arrow[u] \arrow[dl] \arrow[dr] & & \\
& U_a \arrow[uul, "\phi_{ab}"] \arrow[dr, "\phi_{ac}"'] & & U_c \arrow[uur, "\phi_{cb}"'] \arrow[dl, "\phi_{ca}"] & \\
& & U_{ac} & &
\end{tikzcd}$$

Figure 4.2: The triangle of groups corresponding to the KMS group of type $\mathrm{HB}_2^{(2)}$.

We have that $U_a \cong U_b \cong U_c \cong (K, +)$ and

$$U_{ab} \cong \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in K \right\},$$
$$U_{bc} \cong U_{ac} \cong \left\{ \begin{pmatrix} 1 & w & wy+z & y+wx \\ 0 & 1 & y & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -w & 1 \end{pmatrix} \mid w, x, y, z \in K \right\} \leq \mathrm{Sp}_4(K)$$

where $\mathrm{Sp}_4(K) = \{A \in \mathrm{GL}_4(K) \mid A^T J A = J\}$ for $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. The morphisms are as follows

$$\phi_{ab}: U_a \to U_{ab}: x \mapsto \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \phi_{ba}: U_b \to U_{ab}: z \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

$$\phi_{ac}: U_a \to U_{ac}: w \mapsto \begin{pmatrix} 1 & w & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -w & 1 \end{pmatrix} \phi_{ca}: U_c \to U_{ac}: x \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\phi_{bc}: U_b \to U_{bc}: w \mapsto \begin{pmatrix} 1 & w & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -w & 1 \end{pmatrix} \phi_{cb}: U_c \to U_{bc}: x \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

When $K = \mathbb{F}_p$ with p a prime, the KMS-group of type $\mathrm{HB}_2^{(2)}$ has the following presentation:

$$\begin{aligned} \mathcal{U}_A(K) \cong \langle a, b, c \mid\ & a^p, b^p, c^p, [a,b,a], [a,b,b], [c,b,c], [c,b,b,b], [c,b,b,c], \\ & [c,a,c], [c,a,a,a], [c,a,a,c] \rangle. \end{aligned} \tag{4.1}$$

Remark 4.1.3. By the universal property of the direct limit, there exist morphisms $f_J : U_J \to \mathcal{U}_A(K)$ for all $J \in Q_A$. On the other hand, all groups U_J are subgroups of $G_A(K)$. Hence we have canonical morphisms $g_J : U_J \to G_A(K)$ compatible with the embeddings $\phi_{J,L}$. Thus, by the universal property, there exists a unique morphism $\psi : \mathcal{U}_A(K) \to G_A(K)$ such that $g_J = \psi \circ f_J$. In particular, since the g_J are injective, the f_J also need to be injective. By slight abuse of notation, we will denote the image of U_J under f_J in $\mathcal{U}_A(K)$ by U_J again.

Remark 4.1.4. If we equip Q_A with the ordering of reversed inclusion $\subseteq^{\mathrm{op}}$ (i.e. $J \subseteq^{\mathrm{op}} L :\iff J \supseteq L$) then $(Q_A, \subseteq^{\mathrm{op}}, (U_J)_{J\in Q_A}, (\phi_{J,L})_{L\subseteq^{\mathrm{op}} J \in Q_A})$ is a complex of groups in the sense of [BH99, Definition II.12.11]. The KMS group $\mathcal{U}_A(K)$ is then what is called the fundamental group of the complex of groups.

Remark 4.1.5. If additionally $|K| \geq 5$, then by Proposition 3.6.8 we know that for $J \in Q_A, |J| \geq 2$ it holds that U_J is the direct limit of the groups $U_{\{i,j\}}, i, j \in J$ with respect to $U_{\{i\}} \hookrightarrow U_{\{i,j\}}$.

In particular, the KMS group $\mathcal{U}_A(K)$ is the direct limit of $((U_{\{i,j\}})_{i,j\in I}, (U_{\{i\}} \hookrightarrow U_{\{i,j\}})_{i,j\in I})$. Thus we get the following abstract presentation for the KMS groups:

$$\mathcal{U}_A(K) = \left\langle u_\beta(t) \text{ for } t \in K, \beta \in [\alpha_i, \alpha_j]_{\mathbb{N}}, i, j \in I \mid \mathcal{R} \right\rangle$$

where the set of relations $\mathcal{R}$ is defined as:

for all $\{\alpha, \beta\} \subseteq [\alpha_i, \alpha_j]_{\mathbb{N}},\ i, j \in I,\ t, s \in K$:

$$\begin{aligned} u_\alpha(t) u_\alpha(s) &= u_\alpha(s+t), \\ [u_\alpha(t), u_\beta(s)] &= \prod_{\gamma = k\alpha + l\beta \in]\alpha,\beta[_{\mathbb{N}}} u_\gamma \left(C^{\alpha\beta}_{k,l} t^k s^l \right). \end{aligned}$$

The constants $C_{k,l}^{\alpha,\beta}$ are the *structure constants* as in the definition of the Chevalley and Kac–Moody groups.

This also motivates the name *Kac–Moody–Steinberg* groups, since we only consider (part of) the Steinberg relations between the positive root subgroups of the Kac–Moody group.

We fix the following notations: $U_i = U_{\{i\}} \leq \mathcal{U}_A(K)$ for $i \in I$ and $U_{ij} = U_{\{i,j\}}$ for $i, j \in I$. We denote the generators of $U_i = U_{\alpha_i}$ as $u_{\alpha_i}(\lambda)$ or $u_i(\lambda)$ for short. In particular, to distinguish the generators of the KMS group, we will denote them with u, while we denote the generators of the Kac–Moody and Chevalley groups with x.

Example 4.1.6. If we set $K = \mathbb{F}_p$ and $A_{ij} = -1$ for all $i \neq j$, then $\mathcal{U}_A(K)$ has the following presentation:

$$\mathcal{U}_A(K) = \langle u_1, \dots, u_d \mid u_i^p,\ [[u_i, u_j], u_j]\ (i \neq j) \rangle.$$

This example appears in [EJ10, Proposition 7.4] as an example of a Golod-Shafarevich group with property (T) provided $d \geq 6$ and $p > (d-1)^2$.

Corollary 4.1.7. The map $\psi : \mathcal{U}_A(K) \to G_A(K)$ from Remark 4.1.3 given by mapping the generator $u_{\alpha_i}(\lambda)$ of the root subgroup U_i of $\mathcal{U}_A(K)$ to the generator $x_{\alpha_i}(\lambda)$ of $G_A(K)$ has the following properties.

(i) The image of ψ is contained in U^+.

(ii) If A is 2-spherical and $|K| \geq 4$ then the image of ψ is equal to U^+.

(iii) If A is 3-spherical and $|K| \geq 5$ then $\psi : \mathcal{U}_A(K) \to U^+$ is an isomorphism.

Proof. The first point is clear from the definition. The other two points are a direct consequence of Proposition 3.6.8. □

4.2 KMS complexes

In [BH99, Chapter II.12], Bridson and Haefliger associate to a complex of groups certain partially ordered sets (posets) and their geometric realization via "The basic construction". If we apply this construction to the complex of groups $((U_J)_{J \in Q_A}, (\phi_{J,L})_{J \subseteq L \in Q_A})$ (see Remark 4.1.4) and a group G with map $\phi : \mathcal{U}_A(K) \to G$ which is injective on all the U_J the construction gives rise to a poset

$$D(Q_A, \phi) = \bigsqcup_{J \in Q_A} G/\phi(U_J)$$

with ordering $\leq$ given by

$$g\phi(U_J) \leq h\phi(U_L) \iff J \subseteq^{\mathrm{op}} L \text{ and } g^{-1}h \in \phi(U_J).$$

Recall from Remark 2.2.6 that an arbitrary poset can be realized as a simplicial complex by considering its flag complex, but if it satisfies certain properties it can itself be seen as a simplicial complex. We restrict to the following case.

Lemma 4.2.1. If we assume that $Q_A = \mathcal{P}(I) \setminus \{I\}$ and furthermore $\phi(U_J \cap U_L) = \phi(U_J) \cap \phi(U_L)$ then we can identify $D(Q_A, \phi)$ with $\mathcal{CC}(G; (\phi(U_{I\setminus\{i\}}))_{i\in I})$.

Proof. We start with $\mathcal{CC}(G; (\phi(U_{I\setminus\{i\}}))_{i\in I})$. When considering the simplicial complex as a poset, the smallest element corresponds to the empty simplex. One level above that, we have the vertices $\sqcup_{i\in I} G/\phi(U_{I\setminus\{i\}})$. For each $g \in G$ and $i \neq j \in I$ the set $\{g\phi(U_{I\setminus\{i\}}), g\phi(U_{I\setminus\{j\}})\}$ forms an edge. Two such sets $\{g\phi(U_{I\setminus\{i\}}), g\phi(U_{I\setminus\{j\}})\}$, $\{h\phi(U_{I\setminus\{i\}}), h\phi(U_{I\setminus\{j\}})\}$ correspond to the same edge if and only if

$$g^{-1}h \in \phi(U_{I\setminus\{i\}}) \cap \phi(U_{I\setminus\{j\}}) = \phi(U_{I\setminus\{i,j\}}).$$

Hence the set of edges can be identified with the set $\sqcup_{i\neq j\in I} G/\phi(U_{I\setminus\{i,j\}})$. Similarly, the set of k-simplices corresponds to the set $\sqcup_{J\subseteq I:|J|=k+1} G/\phi(U_{I\setminus J})$. Now given a k-simplex represented by $g\phi(U_{I\setminus J})$ its set of vertices is $\{g\phi(U_{I\setminus\{i\}}) \mid i \in J\}$. In particular a vertex $h\phi(U_{I\setminus\{i\}})$ is contained in $g\phi(U_{I\setminus J})$ if and only if $i \in J$ and $g\phi(U_{I\setminus\{i\}}) = h\phi(U_{I\setminus\{i\}})$. But this gives precisely the ordering of the poset $D(Q_A, \phi)$.

The other direction works analogously, with the slight technical trick, that we have to add a smallest element to $\mathcal{P}(I) \setminus \{I\}, \subseteq^{\mathrm{op}}$ which will correspond to the empty simplex. □

This motivates the following definition.

Definition 4.2.2. Let A be a GCM over the index set I which is $(|I|-1)$-spherical but non-spherical, i.e. $Q_A = \mathcal{P}(I) \setminus \{I\}$. Let K be a finite field of size $q \geq 4$. Let $\phi : \mathcal{U}_A(K) \to G$ be a finite quotient of $\mathcal{U}_A(K)$ such that

(i) $\phi|_{U_J}$ is injective for all $J \in Q_A$ (we say ϕ is injective on the local groups),

(ii) $\phi(U_J \cap U_L) = \phi(U_J) \cap \phi(U_L)$ for all $J, L \in Q_A$.

Then we set

$$X = \mathcal{CC}(G; (\phi(U_{I\setminus\{j\}}))_{j\in I})$$

and call X a KMS complex of type A over K.

Since we assume that G is finite, the KMS-complexes will be finite complexes. But we can also apply the construction to the identity map $\mathcal{U}_A(K) \to \mathcal{U}_A(K)$.

Definition 4.2.3. Let A be a GCM over the index set I which is $(|I|-1)$-spherical but non-spherical. Let K be a finite field of size $q \geq 4$. We define the *fundamental KMS complex* of type A over K to be

$$Y_A(K) = \mathcal{CC}(\mathcal{U}_A(K); (U_{I\setminus\{i\}})_{i\in I}).$$

Corollary 4.2.4. Let A be a GCM over the index set I which is $(|I|-1)$-spherical but non-spherical. Let K be a finite field of size $q \geq 5$. If $|I|-1 \geq 3$ then

$$Y_A(K) \cong \Delta^-_{\mathrm{op}}$$

where Δ^-_{op} is the opposite complex of the twin building associated to $G_A(K)$.

Proof. This follows from Corollary 4.1.7, since in this case $U^+ \cong \mathcal{U}_A(K)$ and the isomorphism preserves the local groups. □

Before we have a closer look at the KMS complexes and their properties, we show that finite quotients satisfying the assumptions of Definition 4.2.2 exist.

Finite quotients of KMS groups via residual finiteness Already having high-dimensional expanders in mind, we are not only interested in constructing one finite quotient with the required properties, but an infinite family of finite quotients of increasing size.

We fix a GCM A with rank $n+1$ which is n-spherical. Additionally, we fix a finite field K with at least 4 elements. We want to use that the group U^+ of the Kac–Moody group $G_A(K)$ is residually finite. We start with a technical observation.

Observation 4.2.5. Let $f : X \to Y$ be a homomorphism between two groups X, Y and let $A, B \leq X$ be subgroups such that $f|_A$ and $f|_B$ are injective. Then

$$f(A) \cap f(B) = f(A \cap B) \iff \ker(f) \cap [(A \setminus (A \cap B)) \cdot (B \setminus (A \cap B))] = \emptyset$$

where for any sets $A', B' \subseteq X$ we define $A' \cdot B' := \{a \cdot b \mid a \in A', b \in B'\}$.

Lemma 4.2.6. Let $\psi : \mathcal{U}_A(K) \to U^+$ be as in Remark 4.1.3 and write $\mathbf{U}_J = \psi(U_J)$ for all $J \in Q_A$. Let H be any group and $\phi : U^+ \to H$ a homomorphism satisfying

(i) $\phi|_{\mathbf{U}_J}$ is injective for all $J \in Q_A$;

(ii) $\phi(\mathbf{U}_J) \cap \phi(\mathbf{U}_L) = \phi(\mathbf{U}_J \cap \mathbf{U}_L)$ for all $J, L \in Q_A$.

Then the composition $f = \phi \circ \psi : \mathcal{U}_A(K) \to H$ satisfies, for all $J, L \in Q_A$

$$f(U_J) \cap f(U_L) = f(U_J \cap U_L)$$

and f is injective on the local groups.

Proof. The map f is injective on the local groups as it is the composition of two maps that are injective on the local groups.

Let $J, L \in Q_A$ and set $U'_J = U_J \setminus (U_J \cap U_L)$ and $\mathbf{U}'_J = \mathbf{U}_J \setminus (\mathbf{U}_J \cap \mathbf{U}_L)$ and similarly for U'_L and $\mathbf{U}'_L$. Using Observation 4.2.5 it suffices to show $\ker(f) \cap (U'_J \cdot U'_L) = \emptyset$. Note that $\ker(f) = \ker(\phi \circ \psi) = \psi^{-1}(\ker(\phi))$. Moreover,

$$\psi^{-1}(\ker(\phi)) \cap U'_J \cdot U'_L = \emptyset \iff \ker(\phi) \cap \psi(U'_J \cdot U'_L) = \emptyset. \tag{4.2}$$

Since ψ is a homomorphism injective on the local groups, we have $\psi(U'_J \cdot U'_L) = \mathbf{U}'_J \cdot \mathbf{U}'_L$. By assumption, the equality on the right-hand side of the equivalence in Equation 4.2 holds. □

Note that the sets $\mathbf{U}_J \cdot \mathbf{U}_L$ are finite. Since U^+ is residually finite together with the fact that $|\bigcup_{J,L \in Q_A} \mathbf{U}_J \cdot \mathbf{U}_L| < \infty$, Lemma 4.2.6 implies the following corollary.

Corollary 4.2.7. Let A be a 2-spherical GCM and let K be a finite field of characteristic p such that $|K|\geq 4$. Then, there is an infinite family of finite index normal subgroups $N_1, N_2, \cdots \trianglelefteq_{\text{f.i.}} U^+$ such that

(i) $N_i \cap \left(\bigcup_{J,L\in Q_A} \mathbf{U}_J \cdot \mathbf{U}_L\right) = \{1\}$ for all i, and;

(ii) $\lim_{i\to\infty}|U^+/N_i|=\infty$.

Moreover, the quotient maps

$$\phi_i = \psi \circ \varphi_i : \mathcal{U}_A(K) \xrightarrow{\psi} U^+ \xrightarrow{\varphi_i} U^+/N_i$$

are injective on the local groups and for any $J, L \in Q_A$ we have $\phi_i(U_J) \cap \phi_i(U_L) = \phi_i(U_J \cap U_L)$.

In particular, this infinite family of finite quotients gives rise to an infinite family of KMS complexes.

Structure of the links The key property of the KMS complexes that will make them useful to construct high-dimensional expanders is that we can describe the links explicitly. In particular, they are independent of the finite quotient $\phi : \mathcal{U}_A(K) \to G$.

We fix the following situation: Let A be a GCM over the index set I which is $(|I|-1)$-spherical but non-spherical. Set $n = |I|-1$. Let K be a finite field of size $q \geq 4$. Let $\phi : \mathcal{U}_A(K) \to G$ be a finite quotient of $\mathcal{U}_A(K)$ satisfying the assumptions of Definition 4.2.2. Let $X = \mathcal{CC}(G; \phi(U_{I\setminus\{i\}})_{i\in I})$ denote the corresponding KMS complex. For simplicity, we denote $H_i = \phi(U_{I\setminus\{i\}})$ and $H_T = \cap_{i\in T} H_i$.

Proposition 4.2.8. Let $\sigma \in X(i)$ for $i \leq n-2$ be of type $\emptyset \neq T \subseteq I$. Then

$$\mathrm{lk}_X(\sigma) \cong \mathcal{CC}\left(H_T; \left(H_{T\cup\{i\}}\right)_{i\in I\setminus T}\right) \cong \mathcal{CC}\left(U_{I\setminus T}; \left(U_{I\setminus(T\cup\{i\})}\right)_{i\in I\setminus T}\right).$$

Proof. Firstly, note that, using the assumptions of Definition 4.2.2, we have

$$H_T = \cap_{i\in T}\phi(U_{I\setminus\{i\}}) = \phi(\cap_{i\in T} U_{I\setminus\{i\}}) = \phi(U_{I\setminus T})$$

where the last equality follows from Lemma 3.6.5.

Thus, $\phi|_{U_{I\setminus T}} \colon U_{I\setminus T} \to \phi(U_{I\setminus T}) = H_T$ is an isomorphism of groups that maps $H_{T\cup\{i\}}$ to $U_{I\setminus(T\cup\{i\})}$ and hence induces an isomorphism of coset complexes. □

Corollary 4.2.9. Let X be a KMS complex of type A over K as above and let $\sigma \in X(i)$ for $i \leq n-2$ be of type $\emptyset \neq T \subseteq I$. Then

$$\mathrm{lk}_X(\sigma) \cong \Delta_{\mathrm{op}}$$

where Δ_{op} here denotes the opposite complex in the building associated to $\mathrm{Chev}_{A_{I\setminus T}}(K)$.

Proof. This follows from Proposition 4.2.8 and Proposition 3.6.6. Note that by assumption $A_{I\setminus T}$ is spherical. □

Corollary 4.2.10. Let X be a KMS complex of type A over K as above and let $\sigma \in X(i)$ for $i \leq n-2$ be of type $\emptyset \neq T \subseteq I$. Since A is 2-spherical and K has at least 4 elements all the links $\mathrm{lk}_X(\sigma)$ are connected. Furthermore, X itself is connected.

Proof. This follows from the characterization of connectivity of coset complexes in Lemma 2.4.6 together with the fact that for $\emptyset \neq T \subsetneq I$ we have

$$\begin{aligned}
\langle U_{I\setminus(T\cup\{i\})} : i \in I \setminus T\rangle &= \langle U_{\{j\}} : j \in I \setminus (T \cup \{i\}), i \in I \setminus T\rangle \\
&= \langle U_{\{j\}} : j \in \cup_{i\in I\setminus T} I \setminus (T \cup \{i\})\rangle \\
&= \langle U_{\{j\}} : j \in I \setminus T\rangle = U_{I\setminus T}.
\end{aligned}$$

Since ϕ is surjective, we have $G = \langle \phi(U_{\{i\}}) \mid i \in I\rangle = \langle \phi(U_{I\setminus\{i\}}) \mid i \in I\rangle$. □

Bounded degree When constructing an infinite family of high-dimensional expanders, we want that complexes in the family have a uniform bound on the degree. In this section, we establish that all KMS complexes of type A over a finite field K have the same degree bound, only depending on A and K.

Again, let A be of rank $n+1$, n-spherical and purely n-spherical, K a finite field with at least 4 elements. Let $\phi : \mathcal{U}_A(K) \to G$ denote a finite quotient satisfying the assumptions of Definition 4.2.2 and let $X = \mathcal{CC}(G, (H_i)_{i\in I})$ denote the corresponding KMS complex. Recall that $H_i = \phi(U_{I\setminus\{i\}})$.

We want to consider the number of maximal faces that contain a given arbitrary vertex gH_i.

Maximal faces in X are of the form $\{hH_i \mid i \in I\}, h \in G$. Two maximal faces $\{hH_i \mid i \in I\}$, $\{h'H_i \mid i \in I\}$ coincide if and only if $h^{-1}h' \in \bigcap_{i\in I} H_i = \phi(\bigcap_{i\in I} U_{\{i\}}) = \{1\}$. Thus, every element in G gives rise to a unique maximal face.

Which maximal faces contain a given vertex gH_i? Note that:

$$gH_i \in \{hH_j \mid j \in I\} \iff gH_i = hH_i \iff h \in gH_i.$$

Thus

$$\deg(gH_i) = |H_i| = |U_{I\setminus\{i\}}| = |\langle U_{\{j\}} \mid j \in I \setminus \{i\}\rangle|.$$

The cardinality of $U_{I\setminus\{i\}}$ is equal to $|K|^\ell$ where ℓ is the number of positive roots in the root system of type $A_{I\setminus\{i\}}$ by [Car89, Theorem 5.3.3]. The number of positive roots in the root system of type $A_{I\setminus\{i\}}$ is bounded from above by $(|I|-1)^2 = n^2$ if $A_{I\setminus\{i\}}$ is of classical type (i.e. A_n, B_n, C_n, D_n). Exact numbers, also for the exceptional types, can be found in [Bou08, Chapter 6, Section 4].

Thus the coset complexes we obtain are of bounded degree and the bound depends only on the choice of the generalized Cartan matrix and the field we chose for the Kac–Moody–Steinberg group $\mathcal{U}_A(K)$.

4.3 Congruence KMS complexes

Up to this point, we have proven that KMS complexes exist using the residual finiteness ness of U^+. In this section, we give an explicit construction of KMS complexes using matrix groups over polynomial rings. This construction works for all KMS groups of affine untwisted type. We recall the following setup from Chapter 3.

Let $\mathring{A}$ be a Cartan matrix with root system $\mathring{\Phi}$ and set of simple roots $\mathring{\Pi} = \{\alpha_1, \dots, \alpha_n\}$ and highest root γ, and let A be the corresponding untwisted affine GCM with real roots Φ and simple roots $\Pi = \{\overline{\alpha}_0, \dots, \overline{\alpha}_n\}$. Recall that we use the notation $a_{\alpha,m} = \overline{\alpha} + m\overline{\delta}$ to describe the real roots. Let K be a finite field of size at least 4. Composing the map φ from Proposition 3.4.8 with ψ from Remark 4.1.3 yields the following.

$$\begin{aligned} \phi : \mathcal{U}_A(K) \xrightarrow{\psi} U^+_{\mathring{A}}(K) &\xrightarrow{\varphi} \mathrm{Chev}_{\mathring{A}}(K[t]) \\ u_{\alpha_i}(\lambda) \mapsto x_{\alpha_i}(\lambda) \ &\mapsto \begin{cases} x_{\alpha_i}(\lambda) & i \neq 0 \\ x_{-\gamma}(\lambda t) & i = 0 \end{cases}. \end{aligned}$$

To construct an infinite family of finite quotients we proceed as follows. Let $f \in K[t]$ be an irreducible polynomial of degree $\ell \geq 2$ and let (f) denote the ideal generated by f. We denote by $\pi_f : K[t] \to K[t]/(f)$ the projection to the quotient of the polynomial ring by the ideal generated by f.

Using the functoriality of the Chevalley groups, this induces a map which we again denote by π_f:

$$\begin{aligned} \pi_f : \mathrm{Chev}_{\mathring{A}}(K[t]) &\to \mathrm{Chev}_{\mathring{A}}(K[t]/(f)) \\ x_\alpha(\lambda) &\mapsto x_\alpha(\lambda + (f)). \end{aligned}$$

Our goal is now to show that the composition $\phi_f = \pi_f \circ \phi$

$$\begin{aligned} \phi_f : \mathcal{U}_A(K) \xrightarrow{\phi} \ &\mathrm{Chev}_{\mathring{A}}(K[t]) \quad \xrightarrow{\pi_f} \ \mathrm{Chev}_{\mathring{A}}(K[t]/(f)) \\ u_{\alpha_i}(\lambda) \mapsto &\begin{cases} x_{\alpha_i}(\lambda) & i \neq 0 \\ x_{-\gamma}(\lambda t) & i = 0 \end{cases} \mapsto \begin{cases} x_{\alpha_i}(\lambda + (f)) & i \neq 0 \\ x_{-\gamma}(\lambda t + (f)) & i = 0 \end{cases} \end{aligned}$$

satisfies the assumptions of Definition 4.2.2, i.e. is surjective, injective on the local groups and the intersection of the images of two local groups is the image of the intersection.

Then fixing an infinite family of irreducible polynomials of increasing degree will give rise to an infinite family of KMS complexes of the same type and with increasing size.

For $J \in Q_A$ note that $\{\bar{\alpha}_j \mid j \in J\} \subset \Pi$ and $\{\alpha_j \mid j \in J\} \subset \mathring{\Pi} \cup \{\alpha_0\}$ are sets of linear independent roots that generate roots subsystems $\Phi_J = \bigoplus_{j \in J} \mathbb{Z}\overline{\alpha}_j \cap \Phi$ and $\mathring{\Phi}_J = \bigoplus_{j \in J} \mathbb{Z}\alpha_j \cap \mathring{\Phi}$, respectively. Their set of positive roots is denoted by $\Phi_J^+ = \bigoplus_{j \in J} \mathbb{N}\overline{\alpha}_j \cap \Phi$ and $\mathring{\Phi}_J^+ = \bigoplus_{j \in J} \mathbb{N}\alpha_j \cap \mathring{\Phi}$, respectively.

Remark 4.3.1. Due to the way we defined A from $\mathring{A}$, we have that the angles between $\overline{\alpha}_i, \overline{\alpha}_j$ and α_i, α_j are the same. Thus, for every $J \in Q_A$ we have $\Phi_J \cong \mathring{\Phi}_J$.

Proposition 4.3.2. We have the following description of the image of the local groups under ϕ, where the product is taken in an arbitrary fixed order on Φ_J^+.

$$\phi(U_J) = \left\{ \prod_{a_{\alpha,m}\in\Phi_J^+} x_\alpha(\lambda_{a_{\alpha,m}} t^m) \mid \lambda_{a_{\alpha,m}} \in K \right\}.$$

Proof. Note that, for $J \in Q_A$, U_J is up to isomorphism the unipotent subgroup of the positive standard Borel subgroup of $\mathrm{Chev}_{A_J}(K)$. Thus, from [Car89, Theorem 5.3.3] for height preserving orders and [Ste16, Lemma 16] for arbitrary orders on Φ_J^+, we get that

$$U_J = \left\{ \prod_{\alpha\in\Phi_J^+} u_\alpha(\lambda_\alpha) \mid \lambda_\alpha \in K \right\}.$$

Thus,

$$\begin{aligned}\phi(U_J) &= \left\{ \prod_{\alpha\in\Phi_J^+} \phi(u_\alpha(\lambda_\alpha)) \mid \lambda_\alpha \in K \right\} = \left\{ \prod_{a_{\alpha,m}\in\Phi_J^+} \phi(u_{a_{\alpha,m}}(\lambda_{a_{\alpha,m}})) \mid \lambda_{a_{\alpha,m}} \in K \right\} \\ &= \left\{ \prod_{a_{\alpha,m}\in\Phi_J^+} x_\alpha(\lambda_{a_{\alpha,m}} t^m) \mid \lambda_{a_{\alpha,m}} \in k \right\}.\end{aligned}$$

□

Remark 4.3.3. Let $a_{\alpha,m} \in \Phi_J^+$ for some $J \in Q_A$. If $0 \notin J$, then we can write $a_{\alpha,m} = \overline{\alpha} + m\overline{\delta} = \sum_{j\in J} n_j\overline{\alpha}_j = \overline{\sum_{j\in J} n_j\alpha_j}$ for some $n_j \in \mathbb{N}$ and thus $\alpha = \sum_{j\in J} n_j\alpha_j$ and $m = 0$.

If $0 \in J$, then, for some $n_j \in \mathbb{N}$, we have

$$a_{\alpha,m} = \sum_{j\in J} n_j\overline{\alpha}_j = \overline{\underbrace{\sum_{j\in J\setminus\{0\}} n_j\alpha_j}_{=\beta\in\mathring{\Phi}}} + n_0\overline{\alpha}_0 = \overline{\beta} + \overline{-\gamma} + \overline{\delta} = \overline{\beta - \gamma} + \overline{\delta}$$

and thus $m = 1$.

Lemma 4.3.4. Let $f \in K[t]$ be an irreducible polynomial with $\deg(f) \geq 2$. Then the map $\phi_f = \pi_f \circ \phi : \mathcal{U}_A(K) \to \mathrm{Chev}_{\mathring{A}}(K[t]/(f))$ is injective on the local groups $U_J, J \in Q_A$.

Proof. We know that ϕ is injective on the local groups as a composition of maps which are injective on the local groups. Since $\deg(f) \geq 2$ we have that $\pi_f|_K$ and $\pi_f|_{K\cdot t}$ are injective. The results of Proposition 4.3.2 and Remark 4.3.3 yield that the arguments inside the x_α in the image of a local group are polynomials of degree less or equal to 1. □

Next, we show that ϕ_f satisfies that the image of the intersection of two local groups is the intersection of the images.

Lemma 4.3.5. There exists $\ell \in \mathbb{N}$, depending only on the GCM A and K, such that for any irreducible polynomial $f \in K[t]$ with $\deg(f) \geq \ell$ and for all $J, L \in Q_A$, we have

$$\phi_f(U_J) \cap \phi_f(U_L) = \phi_f(U_J \cap U_L).$$

Proof. We know that in $\phi(U_J), \phi(U_L)$ the degree of the arguments inside the x_α of Proposition 4.3.2 is at most one. Since the sets U_J, U_L are finite, we know that the degree of t inside the x_α that appear in $\phi(U_J) \cdot \phi(U_L)$ is bounded above by some constant $\ell_{J,L}$ depending on the size and the type of J, L. Set $\ell = \max_{J,L \in Q_A} \ell_{J,L} + 1$. Then, by choosing f of degree at least ℓ, we ensure that π_f is injective on $\phi(U_J) \cdot \phi(U_L)$ for all $J, L \in Q_A$.

Furthermore, we know that the map φ from U^+ to $\mathrm{Chev}_{\mathring{A}}(K[t])$ is injective. Thus $\pi_f \circ \varphi$ is injective on $\bigcup_{J,L \in Q_A} \psi(U_J) \cdot \psi(U_L)$. Hence, Lemma 4.2.6 gives the desired result. $\square$

For fields of characteristic different from 2, this result can be made substantially more precise using the adjoint representation of $\mathrm{Chev}_{\mathring{A}}(K[t]/(f))$. For the characteristic 2 case, see also Remark 4.3.7.

Proposition 4.3.6. Let K be a finite field of characteristic different from 2, $|K| \geq 5$ and let $f \in K[t]$ be irreducible with $\deg(f) \geq 2$. Then for all $J, L \in Q_A$, we have

$$\phi_f(U_J) \cap \phi_f(U_L) = \phi_f(U_J \cap U_L).$$

Proof. Recall that

$$\begin{aligned} \phi_f : \mathcal{U}_A(K) \xrightarrow{\phi} \mathrm{Chev}_{\mathring{A}}(K[t]) \quad &\xrightarrow{\pi_f} \mathrm{Chev}_{\mathring{A}}(K[t]/(f)) \\ u_i(\lambda) \mapsto \begin{cases} x_{\alpha_i}(\lambda) & i \neq 0 \\ x_{-\gamma}(\lambda t) & i = 0 \end{cases} &\mapsto \begin{cases} x_{\alpha_i}(\lambda + (f)) & i \neq 0 \\ x_{-\gamma}(\lambda t + (f)) & i = 0 \end{cases}. \end{aligned}$$

We show that the map π_f is injective on $\phi(U_J) \cdot \phi(U_L)$. To do so, we consider the adjoint representation of the Chevalley group $\mathrm{Ad} : \mathrm{Chev}_{\mathring{A}}(K[t]/(f)) \to \mathrm{GL}(\mathfrak{g})$ and show that the composition $\mathrm{Ad} \circ \pi_f$ is injective on $\phi(U_J) \cdot \phi(U_L)$. Here $\mathfrak{g}$ denotes the Lie algebra of $G = \mathrm{Chev}_{\mathring{A}}(K[t]/(f))$.

We use the notation and results from Section 3.2. By [Mar18, Theorem 7.55] there exists a maximal split torus T of G such that $\ker(\mathrm{Ad}) \subseteq T$. Furthermore, we have $\pi_f(\phi(U_J)) \cap T = \pi_f(\phi(U_L)) \cap T = \{1\}$. Hence, Ad is injective on $\pi_f(\phi(U_J))$ and $\pi_f(\phi(U_L))$.

Next, we show that $\ker(\mathrm{Ad}) \cap \pi_f(\phi(U_J) \cdot \phi(U_L)) = \{1\}$. Suppose there exists a $1 \neq z \in \pi_f(\phi(U_J) \cdot \phi(U_L))$ that acts trivially on $\mathfrak{g}$, i.e. $\mathrm{Ad}(z) = 1$. Then there exist $g \in \pi_f(\phi(U_J)), g' \in \pi_f(\phi(U_L))$ such that $\mathrm{Ad}(g) = \mathrm{Ad}(g')$. Without loss of generality, we can assume that $J = I \setminus \{i\}$ for some $i \in I$ (otherwise $J \subset I \setminus \{i\}$ for some $i \in I$, thus $U_J \subseteq U_{I \setminus \{i\}}$ and hence $\pi_f(\phi(U_J)) \subseteq \pi_f(\phi(U_{I \setminus \{i\}})))$. Assuming that $L \nsubseteq J$ (otherwise the statement is trivial) we get that $i \in L$.

Let $\mathfrak{L}_J$ be the subspace of $\mathfrak{g}$ spanned by the basis $\{h_\alpha, e_\beta \mid \alpha \in \mathring{\Pi}, \beta \in \mathring{\Phi}_J^+\}$. Then $\mathfrak{L}_J$ is a Lie subalgebra of $\mathfrak{g}$. Note that $\mathrm{Ad}(g)$ stabilizes $\mathfrak{L}_J$ and that, if we define $\mathfrak{L}_L$ analogously to $\mathfrak{L}_J$, we have that $\mathrm{Ad}(g')$ stabilizes $\mathfrak{L}_L$.

Next, we show that there exists a $h \in \mathfrak{h}$ with $\mathrm{Ad}(g')(h) \notin \mathfrak{L}_J$. Note that any $\alpha \in \mathring{\Phi}_L^+$ has a unique decomposition as $\sum_{i \in L} n_i \alpha_i$ for some $n_i \in \mathbb{N}$. We consider the height of α in $\mathring{\Phi}_L^+$ i.e. $\mathrm{ht}(\alpha) = \sum_{i \in L} n_i$. We order the roots in $\mathring{\Phi}_L^+$ in the following way: we start with the roots in $\mathring{\Phi}_L^+ \setminus \mathring{\Phi}_J^+$ and order them increasing in height. Afterwards, we arrange the remaining roots, which are in $\mathring{\Phi}_L^+ \cap \mathring{\Phi}_J^+$, in arbitrary order. Then g' can be written in the following way, where the product is taken in the order that we just fixed (see Proposition 4.3.2):

$$g' = \prod_{\alpha \in \mathring{\Phi}_L^+} x_\alpha(s_\alpha).$$

Here the s_α are elements of $K[t]/(f)$. Let $\alpha \in \mathring{\Phi}_L^+$ be the first root in the order such that $s_\alpha \neq 0$. Since $g' \notin \pi_f(\phi(U_J))$, we have that $\alpha \in \mathring{\Phi}_L^+ \setminus \mathring{\Phi}_J^+$.

The set $P := \mathring{\Phi}_L^+ \setminus \mathring{\Phi}_J^+$ has the following two properties. For all $\beta, \gamma \in P$ such that $\beta + \gamma \in \mathring{\Phi}^+$ we have $\beta + \gamma \in P$ (we say that P is *closed*), and for all $\beta \in P, \gamma \in \mathring{\Phi}_L^+$ with $\beta + \gamma \in \mathring{\Phi}^+$ we have $\beta + \gamma \in P$ (we say that P is an ideal in $\mathring{\Phi}_L^+$).

We inductively apply formulas (3.5) and (3.7) to $\mathrm{Ad}(g')(h_\alpha) = \mathrm{Ad}(\prod_{\beta \in \mathring{\Phi}_L^+} x_\beta(s_\beta))(h_\alpha)$, using the observations above and that $x_\alpha(s_\alpha)$ is the first non-trivial term in g'. Further using that $\mathrm{Ad}(x_\alpha(s_\alpha))(h_\alpha) = h_\alpha - 2s_\alpha e_\alpha$, we have that

$$\mathrm{Ad}(g')(h_\alpha) = h_\alpha - 2s_\alpha e_\alpha + \sum_{\beta \in \mathring{\Phi}_J^+ \cap \mathring{\Phi}_L^+} \lambda_\beta e_\beta + \sum_{\gamma \in \mathring{\Phi}_L^+ \setminus \mathring{\Phi}_J^+ :\ \mathrm{ht}(\gamma) \geq \mathrm{ht}(\alpha), \gamma \neq \alpha} \lambda_\gamma e_\gamma$$

for some $\lambda_\beta, \lambda_\gamma \in K[t]/(f)$.

Since we assume $\mathrm{char}(K) \neq 2$, we have $\mathrm{Ad}(g')(h_\alpha) \notin \mathfrak{L}_J$, a contradiction to $\mathrm{Ad}(g) = \mathrm{Ad}(g')$. Therefore, we have shown that $\ker(\mathrm{Ad}) \cap \pi_f(\phi(U_J)\phi(U_L)) = \{1\}$.

Now, we can conclude that π_f is injective on $\phi(U_J) \cdot \phi(U_L)$. Let $1 \neq a \in \phi(U_J), 1 \neq b \in \phi(U_L)$ such that $ab \neq 1$. We want to show that $\pi_f(ab) \neq 1$. By Lemma 4.3.4, we have $\pi_f(a) \neq 1, \pi_f(b) \neq 1$. Since both are in $\pi_f(\phi(U_J) \cdot \phi(U_L))$ and $\pi_f(\phi(U_J) \cdot \phi(U_L)) \cap \ker(\mathrm{Ad}) = \{1\}$, we know $\mathrm{Ad}(\pi_f(a)) \neq 1, \mathrm{Ad}(\pi_f(b)) \neq 1$ and $\mathrm{Ad}(\pi_f(a)\pi_f(b)) = \mathrm{Ad}(\pi_f(ab)) \neq 1$. This implies that $\pi_f(ab) \neq 1$.

Furthermore, we know that $\varphi : U^+ \to \mathrm{Chev}_{\mathring{A}}(K[t])$ is injective. Thus $\pi_f \circ \varphi$ is injective on $\bigcup_{J,L \in Q_A} \psi(U_J) \cdot \psi(U_L)$. Hence, Lemma 4.2.6 gives the desired result. □

Remark 4.3.7. If the characteristic of K is 2, the proof of Proposition 4.3.6 only fails if $\alpha(h) = 0$ for all h in the Cartan subalgebra. The root α is the image of a simple root, say α_i, under the action of an element of the Weyl group. Thus,

$$\alpha(h) = 0 \text{ for all } h \in \mathfrak{h} \iff \alpha_i(h) = 0 \text{ for all } h \in \mathfrak{h} \iff \mathring{A}_{ji} = \alpha_i(h\alpha_j) = 0 \text{ for all } j \in \mathring{I}$$

If the Cartan matrix $\mathring{A}$ has no column equal to 0 mod 2, e.g. the Dynkin diagram has no double edges, then the above proof still works by choosing h such that $\alpha(h) \neq 0$ in K instead of taking h_α. Hence, if the characteristic of K is 2 and the Cartan matrix is not of type C_n, we still get the result of Proposition 4.3.6.

Lastly, we show that ϕ_f is surjective.

Proposition 4.3.8. The image of ϕ has the following form:

$$\phi(\mathcal{U}_A(K)) = \left\langle x_\alpha(g) \mid \alpha \in \mathring{\Phi}, g \in K[t] \text{ if } \alpha \in \mathring{\Phi}^+, g \in (t) \lhd K[t] \text{ otherwise } \right\rangle .$$

Proof. Recall that $\Phi^+ = \{\overline{\alpha} + m\overline{\delta} \mid m = 0 \text{ and } \alpha \in \mathring{\Phi}^+ \text{ or } m \geq 1, \alpha \in \mathring{\Phi}\}$ and for $u_{a_{\alpha,m}}(\lambda) \in U^+$ we have $\varphi(u_{a_{\alpha,m}}(\lambda)) = x_\alpha(\lambda t^m)$. Thus,

$$\begin{aligned}\phi(\mathcal{U}_A(K)) &= \varphi(U^+) = \varphi(\langle u_{a_{\alpha,m}}(\lambda_{a_{\alpha,m}}) \mid a_{\alpha,m} \in \Phi^+, \lambda \in K\rangle) \\ &= \langle x_\alpha(\lambda_\alpha t^m) \mid m = 0 \text{ and } \alpha \in \mathring{\Phi}^+ \text{ or } m \geq 1, \alpha \in \mathring{\Phi}\rangle\end{aligned}$$

which implies the result. □

Corollary 4.3.9. The map $\phi_f = \pi_f \circ \phi : \mathcal{U}_A(K) \to \mathrm{Chev}_{\mathring{A}}(K[t]/(f))$ is surjective.

Proof. This follows from Proposition 4.3.8 together with the fact that $K[t] = (t) + (f)$ and thus we have $\pi_f(K[t]) = \pi_f((t))$. □

The results now justify the following definition.

Definition 4.3.10 (Congruence KMS complexes)**.** Let $\mathring{\Phi}$ be a spherical root system of type $\mathring{A}$ and rank n, let K be a finite field of order ≥ 5 and characteristic different from 2, and let $f \in K[t]$ be an irreducible polynomial of degree ≥ 2. We define the *congruence KMS complex* $X(\mathring{\Phi}, K, f)$ as

$$X(\mathring{\Phi}, K, f) = \mathcal{CC}(\mathrm{Chev}_{\mathring{A}}(K[t]/(f)); (\pi_f(H_i))_{i \in \{0,\dots,n\}})$$

where $\pi_f : \mathrm{Chev}_{\mathring{A}}(K[t]) \to \mathrm{Chev}_{\mathring{A}}(K[t]/(f))$ is the projection induced by the map $K[t] \to K[t]/(f)$. Furthermore, for $i = 1, \dots, n$ we set

$$H_i = \langle x_{-\gamma}(\lambda t), x_{\alpha_j}(\lambda) \mid j \in \{1, \dots, n\} \setminus \{i\}, \lambda \in K\rangle \leq \mathrm{Chev}_{\mathring{A}}(K[t])$$

and

$$H_0 = \langle x_{\alpha_j}(\lambda) \mid j \in \{1, \dots, n\}, \lambda \in K\rangle \leq \mathrm{Chev}_{\mathring{A}}(K[t]).$$

Given a sequence of irreducible polynomials $f_s \in K[t]$ of degree $\ell \geq 2$ such that $\deg(f_s)$ tends to infinity, we define the family of congruence KMS complexes associated to the sequence $\{f_s\}$ to be the family $\{X^{(s)} = X(\mathring{\Phi}, K, f_s)\}_s$.

Corollary 4.3.11. A congruence KMS complex $X(\mathring{\Phi}, K, f)$ is a KMS complex of type A over K with respect to the finite quotient ϕ_f.

Example case: SL_3 In this section, we describe the situation in the case where $\mathring{\Phi}$ is of type A_2, i.e. $\mathrm{Chev}_{\mathring{A}} = \mathrm{SL}_3$. This will give rise to probably the most basic example of a KMS complex.

In this case the generators $x_\alpha(\lambda)$ for $\alpha \in \mathring{\Phi}$ of the Chevalley group can be realized as the following matrices:

$$x_{\alpha_1}(\lambda) = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{\alpha_2}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{-\gamma}(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & 0 & 1 \end{pmatrix}.$$

We consider the map ϕ as described above:

$$\begin{gathered}\phi : \mathcal{U}_A(K) = \langle u_i(\lambda) \mid i = 0,1,2, \lambda \in K \rangle \to \mathrm{Chev}_{\mathring{A}}(K[t]) = \langle x_\alpha(g) \mid \alpha \in \mathring{\Phi}, g \in K[t] \rangle \\ u_1(\lambda) \mapsto x_{\alpha_1}(\lambda) \qquad u_2(\lambda) \mapsto x_{\alpha_2}(\lambda) \qquad u_0(\lambda) \mapsto x_{-\gamma}(\lambda t).\end{gathered}$$

We get the following images of the local groups of the spherical subsets of $I = \{0,1,2\}$:

$$\phi(U_{12}) = \begin{pmatrix} 1 & K & K \\ 0 & 1 & K \\ 0 & 0 & 1 \end{pmatrix}, \qquad \phi(U_{01}) = \begin{pmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ Kt & Kt & 1 \end{pmatrix}, \qquad \phi(U_{02}) = \begin{pmatrix} 1 & 0 & 0 \\ Kt & 1 & K \\ Kt & 0 & 1 \end{pmatrix}.$$

Next, we fix a family of irreducible polynomials $(f_m)_{m\in\mathbb{N}}$ with $\deg(f_m) \geq 2$ for all $m \in \mathbb{N}$, and with $\lim_{m\to\infty} \deg(f_m) = \infty$. Note that when $\mathring{\Phi}$ is of type A_n, then the constants $C_{\alpha,\beta}$ appearing in the proof of Proposition 4.3.6 have value ± 1 and thus do not vanish when seen in $K[t]/(f_m)$ even if K has characteristic 2. We obtain the maps

$$\phi_m = \pi_{f_m} \circ \phi : \mathcal{U}_A(K) \to \mathrm{Chev}_{\mathring{A}}(K[t]/(f_m))$$

by taking each entry of the matrices modulo the ideal generated by f_m.

Setting

$$G_m = \mathrm{Chev}_{\mathring{A}}(K[t]/(f_m)),\ H_0^m = \pi_{f_m}(\phi(U_{12})),\ H_1^m = \pi_{f_m}(\phi(U_{02})),\ H_2^m = \pi_{f_m}(\phi(U_{01}))$$

we get an infinite family of congruence KMS complexes of the form

$$X_m = \mathcal{CC}(G_m; (H_0^m, H_1^m, H_2^m)),\ m \in \mathbb{N}.$$

KMS complexes of fixed type in matrix groups of arbitrary large rank Another example of a KMS complex was given in [Cor25]. There, the author considers the GCM $A = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix}$ with corresponding Dynkin diagram as shown in Figure 4.1.

We denote the simple roots by $\Pi = \{\alpha_a, \alpha_b, \alpha_c\}$ as indicated in Figure 4.1.

Recall that when $K = \mathbb{F}_p$ with p a prime, the KMS-group of type A has the following presentation:

$$\begin{aligned}\mathcal{U}_A(K) \cong \langle a,b,c \mid\ & a^p, b^p, c^p, [a,b,a], [a,b,b], [c,b,c], [c,b,b,b], [c,b,b,c], \\ & [c,a,c], [c,a,a,a], [c,a,a,c] \rangle.\end{aligned} \tag{4.3}$$

The construction of the finite quotient is as follows.
Let $M_a, M_b, M_c \in \mathrm{Mat}_n(K)$. For all $\lambda \in K$, set

$$V_a(\lambda) := \begin{pmatrix} 1 & 0 & 0 & \lambda M_a \\ 0 & 1 & \lambda M_a & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, V_b(\lambda) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ \lambda M_b & 1 & 0 & 0 \\ 0 & 0 & 1 & -\lambda M_b \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$V_c(\lambda) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \lambda M_c & 0 & 1 \end{pmatrix}.$$

Then the assignment

$$u_{\alpha_a}(\lambda) \mapsto V_a(\lambda), \quad u_{\alpha_b}(\lambda) \mapsto V_b(\lambda), \quad u_{\alpha_c}(\lambda) \mapsto V_c(\lambda),$$

induces a group homomorphism

$$\phi_{M_a,M_b,M_c}\colon \mathcal{U}_A(K) \to \mathrm{SL}_{4n}(K).$$

Proposition 4.3.12 ([Cor25, Corollary 6.4])**.** Let $p > 2$ and $k > 3$ be distinct primes. Set $q = p^r$ for some $r \geq 1$. Then there exist matrices $M_a, M_b, M_c \in \mathrm{GL}_k(\mathbb{F}_q)$ such that the map $\phi_{M_a,M_b,M_c} : \mathcal{U}_A(K) \to \mathrm{SL}_{4k}(\mathbb{F}_q)$ satisfies the assumptions of Definition 4.2.2. In particular, setting $H_i = \phi_{M_a,M_b,M_c}(U_{I\setminus\{i\}}), i \in I = \{a, b, c\}$, the complex

$$X_{q,k} = \mathcal{CC}(\mathrm{SL}_{4k}(\mathbb{F}_q); (H_i)_{i\in I})$$

is a KMS complex of type A over $\mathbb{F}_q$. Letting k vary over the set of all primes $k > 3$ with $k \neq p$, we get an infinite family of KMS complexes of type A over $\mathbb{F}_q$.

Proposition 4.3.13 ([Cor25, Corollary 6.5])**.** Let $p > 2$ and $k > 3$ be distinct primes. Set $q = p^r$ for some $r \geq 1$. Then there exist matrices $M_a, M_b, M_c \in \mathrm{GL}_k(\mathbb{F}_q)$ such that the map $\phi_{M_a,M_b,M_c} : \mathcal{U}_A(K) \to \mathrm{SL}_{4k}(\mathbb{F}_q)$ has image $\mathrm{Sp}_{4k}(\mathbb{F}_q)$ and when restricted to that it satisfies the assumptions of Definition 4.2.2. In particular, setting $H_i = \phi_{M_a,M_b,M_c}(U_{I\setminus\{i\}}), i \in I = \{a, b, c\}$, the complex

$$X'_{q,k} = \mathcal{CC}(\mathrm{Sp}_{4k}(\mathbb{F}_q); (H_i)_{i\in I})$$

is a KMS complexe of type A over $\mathbb{F}_q$. Letting k vary over the set of all primes $k > 3$ with $k \neq p$, we get an infinite family of KMS complexes of type A over $\mathbb{F}_q$.

The main difference of this construction compared to the congruence KMS construction is that here we get an infinite family by keeping the field fixed and varying the size of the matrices, while for the congruence KMS complexes we fix the size of the matrices and vary the size of the field.

4.4 Spectral expansion of KMS complexes

For this section, we fix a GCM $A = (A_{ij})_{i,j\in I}$ of rank $|I|= n+1$ which is non-spherical but n-spherical and a finite field K with at least 4 elements. Further, we fix a KMS complex $X = \mathcal{CC}(G; (H_i)_{i\in I})$ with G a finite quotient of $\mathcal{U}_A(K)$ and $H_i = \phi(U_{I\setminus\{i\}})$ as usual.

Our goal is to show that X is a local spectral expander using Theorem 2.3.6. We have already seen that X and every link of a face of dimension $\leq n-2$ is connected. Hence it remains to bound the second largest eigenvalue of the random walk matrices of the links of faces of dimension $n-2$, i.e. those for which the links are graphs.

Let $\sigma \in X(n-2)$ of type $T \subseteq I$. Then $|T|= n-1$ and thus $I \setminus T = \{i, j\}$ has two elements. From Proposition 4.2.8 we get that

$$\mathrm{lk}_X(\sigma) \cong \mathcal{CC}(U_{ij}; (U_i, U_j)).$$

The idea for proving the expansion of the graphs $\mathcal{CC}(U_{ij}; (U_i, U_j))$ is by considering them as $|K|$-regular induced subgraphs of the spherical building of the corresponding type. This is indeed possible since U_{ij} is the positive unipotent subgroup in the Chevalley group of type $A_{\{i,j\}}$ and using Section 3.6. The corresponding spherical building is $|K| + 1$-regular and its spectrum is known. Hence the following lemma gives the desired bound on the second largest eigenvalue λ_2 of the random walk matrix of $\mathcal{CC}(U_{ij}; (U_i, U_j))$. The proof of the lemma only uses basic linear algebra tools and can be found in [HS19, Lemma 5.5].

Lemma 4.4.1. Let X be a d-regular, simple graph that is an induced subgraph of a D-regular graph Y. Then

$$\lambda_2(X) \leq \frac{D}{d}\lambda_2(Y).$$

Motivated by this, we investigate the spectrum of the spherical buildings corresponding to the Chevalley groups of type A_2, B_2, G_2 which cover the cases where i, j are connected by a single, double or triple edge in the Dynkin diagram, respectively.

The eigenvalues of the Laplacian matrix $\Delta = \mathrm{id} - M$ (where M is the random walk matrix) of the spherical building of rank 2 are computed in [Gar73, Proposition 7.10]. Note that in Garlands notation we have $q' = q(1) = q(2) = |K|$ (since by definition $1 + q(i) = |P_{\{i\}}/B| = \deg(1P_{\{i\}}) = |K| + 1$). Hence we can deduce the following values for the spectrum of the random walk matrix depending on the type.

The A_2 case Let $K = \mathbb{F}_q$ and let M be the random walk matrix of the spherical building of type A_2 over K. Then the spectrum is

$$\mathrm{spec}(M) = \left\{1, \frac{\sqrt{q}}{q+1}, -\frac{\sqrt{q}}{q+1}, -1\right\}$$

and hence

$$\lambda_2(\mathcal{CC}(U_{i,j}; (U_i, U_j))) \leq \frac{q+1}{q} \cdot \frac{\sqrt{q}}{q+1} = \frac{1}{\sqrt{q}}.$$

The B_2 case Let $K = \mathbb{F}_q$ and let M be the random walk matrix of the spherical building of type B_2 over K. Then the spectrum is

$$\mathrm{spec}(M) = \{1, \frac{\sqrt{2q}}{q+1}, 0, -\frac{\sqrt{2q}}{q+1}, -1\}$$

and hence

$$\lambda_2(\mathcal{CC}(U_{ij}; (U_i, U_j))) \leq \frac{(q+1)\sqrt{2q}}{q(q+1)} = \sqrt{\frac{2}{q}}.$$

The G_2 case Let $K = \mathbb{F}_q$ and let M be the random walk matrix of the spherical building of type G_2 over K. Then the spectrum is

$$\mathrm{spec}(M) = \{1, \frac{\sqrt{3q}}{q+1}, \frac{\sqrt{q}}{q+1}, 0, -\frac{\sqrt{q}}{q+1}, -\frac{\sqrt{3q}}{q+1}, -1\}$$

and hence

$$\lambda_2(\mathcal{CC}(U_{i,j}; (U_i, U_j))) \leq \frac{(q+1)\sqrt{3q}}{q(q+1)} = \sqrt{\frac{3}{q}}.$$

Remark 4.4.2. If q is prime, the bounds on $\lambda_2(\mathcal{CC}(U_{ij}; U_i, U_j))$ are sharp in the A_2 and B_2 case. This can be seen using the concept of representation angle and the results of [Cap+22, Proposition 7.3].

We can now deduce the first high-dimensional expansion result for KMS-complexes.

Theorem 4.4.3. Let $A = (A_{ij})_{i,j \in I}$ be GCM of rank $n + 1 \geq 3$ which is non-spherical but n-spherical. Let $K = \mathbb{F}_q$ be a finite field of size at least 4. We define the parameter γ depending on A and K as follows:

- If the Dynkin diagram associated to A has at most single edges, we set $\gamma = \frac{1}{\sqrt{q}}$;
- if the Dynkin diagram associated to A has at most double edges, we set $\gamma = \sqrt{\frac{2}{q}}$;
- if the Dynkin diagram associated to A has a triple edge, we set $\gamma = \sqrt{\frac{3}{q}}$.

If $\gamma \leq \frac{1}{n}$ then any KMS complex of type A over K is a γ'-local spectral expander, for $\gamma' = \frac{\gamma}{1-(n-1)\gamma}$.

Proof. Note that if $A_{ij} = 0$ for some $i, j \in I$, i.e. the corresponding vertices in the Dynkin diagram are not adjacent, then $\mathcal{CC}(U_{i,j}; (U_i, U_j))$ is the complete bipartite graph $K_{q,q}$ since the elements of U_i and U_j commute. Hence its second largest eigenvalue is 0. The result then follows from the observations above for the other cases, together with the local-to-global result Theorem 2.3.6 and Corollary 4.2.10, which assures connectivity of the full complex. □

Using the discussion above on the degree of KMS complexes, we deduce the following corollary.

Corollary 4.4.4. Let A and K be as in Theorem 4.4.3 above satisfying $\gamma \leq \frac{1}{n}$. Let $X^{(s)}, s \in \mathbb{N}$ be an infinite family of KMS complexes of type A over K such that $|X^{(s)}(0)| \to \infty$ as $s \to \infty$. Then $(X^{(s)})_{s\in\mathbb{N}}$ is a family of bounded degree local spectral expanders in the sense of Definition 2.3.1.

4.5 Comparison to previous constructions

In this section, we compare the KMS complexes to the previous constructions of high-dimensional expanders that we described in Section 2.5.

Ramanujan complexes For the Ramanujan complexes, all proper links are spherical buildings, while for the KMS complexes the links are opposite complexes in spherical buildings. Thus, locally we consider a substructure which is still large enough to preserve the good connectedness properties of the buildings. An advantage of our construction is that it can be described in terms of coset complexes. There is some similarity between how congruence subgroups are used to construct finite quotients for the Ramanujan complexes and the congruence KMS complexes.

Kaufman-Oppenheim complexes The congruence KMS complexes of type $\tilde{A}_n$ are in some sense similar to the Kaufman-Oppenheim complexes. One difference is that the congruence KMS complexes are over $\mathrm{SL}_n(K[t]/(f))$ where f is an irreducible polynomial and hence $K[t]/(f)$ is a field, while Kaufman and Oppenheim consider $\mathrm{EL}_n(K[t]/(t^s))$. They consider the subgroups

$$K_i = \langle e_{j,j+1}(a+bt) \mid j \in \{1,\ldots,n\} \setminus \{i\}\rangle$$

while for the congruence KMS complexes we take

$$H_i = \left\langle e_{j,j+1}(at^{\epsilon_j}) \mid j \in \{1,\ldots,n\} \setminus \{i\}, \epsilon_j = \begin{cases} 1 & \text{if } j = n \\ 0 & \text{else} \end{cases} \right\rangle.$$

Thus the positions of the entries of the matrices that we fill are the same, but in the KMS case the entries are either filled with field elements or multiples of t instead of polynomials of degree ≤ 1.

This observation does not quite generalize to the complexes in [OP22] for other types. There, the authors quotient out by irreducible polynomials as in the congruence KMS complex construction, but the positions of the entries of the matrices are slightly different. O'Donnell and Pratt define, for a spherical root system Φ (not of type G_2) with simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$, the set $\mathcal{S} = \Pi \cup \{\sum_i \alpha_i\}$ and consider the following subgroups of $\mathrm{Chev}_A(K[t]/(f))$, for $\alpha \in \mathcal{S}$,

$$K_\alpha = \langle x_\beta(a+bt) \mid \beta \in \mathcal{S} \setminus \{\alpha\}, a, b \in K\rangle.$$

We can describe the congruence KMS complexes by setting $\tilde{\mathcal{S}} = \Pi \cup \{-\gamma\}$ where γ is the highest root in Φ (which in type A_n is equal to $\sum_i \alpha_i$ but not in the other cases). Then we consider the subgroups, for $\alpha \in \tilde{\mathcal{S}}$,

$$H_\alpha = \left\langle x_\beta(at^{\epsilon_\beta}) \mid a,b \in K, \beta \in \tilde{\mathcal{S}} \setminus \{\alpha\}, \epsilon_\beta = \begin{cases} 1 & \text{if } \beta = -\gamma \\ 0 & \text{else} \end{cases} \right\rangle.$$

In [OP22] it is mentioned already that also different choices for the set $\mathcal{S}$ are possible. Thus the constructions are similar, but not quite the same.

The complexes from [DLZ23] have the same local structure as the congruence KMS complexes of type A_n in the sense that the proper links are isomorphic. Furthermore, the underlying infinite complexes have isomorphic connected components. It is however not clear if the finite quotients are isomorphic, but they seem to be slightly different.

In both cases the vertex links are isomorphic to

$$\mathcal{CC}\left(\begin{pmatrix} 1 & K & K \\ 0 & 1 & K \\ 0 & 0 & 1 \end{pmatrix}; \left(\begin{pmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & K \\ 0 & 0 & 1 \end{pmatrix}\right)\right).$$

For the underlying infinite complex, consider $\varphi_1 : \mathrm{SL}_3(K[t,t^{-1}]) \to \mathrm{SL}_3(K[t,t^{-1}])$ induced by the map $K[t,t^{-1}] \to K[t,t^{-1}] : t \mapsto t^3$. Further, consider $\varphi_2 : \mathrm{SL}_3(K[t,t^{-1}]) \to \mathrm{SL}_3(K[t,t^{-1}])$ given by conjugation with the matrix

$$T = \begin{pmatrix} t & & \\ & 1 & \\ & & t^{-1} \end{pmatrix}.$$

Then $\varphi = \varphi_2 \circ \varphi_1$ is an injective homomorphism of $\mathrm{SL}_3(K[t,t^{-1}])$ to itself that maps the subgroups

$$H_1 = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a,b,c \in K \right\},\ H_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ ct & 1 & a \\ bt & 0 & 1 \end{pmatrix} \mid a,b,c \in K \right\},$$
$$H_3 = \left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ bt & ct & 1 \end{pmatrix} \mid a,b,c \in K \right\}$$

from the congruence KMS construction to

$$K_1 = \left\{ \begin{pmatrix} 1 & at & ct^2 \\ 0 & 1 & bt \\ 0 & 0 & 1 \end{pmatrix} \mid a,b,c \in K \right\},\ K_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ ct^2 & 1 & at \\ bt & 0 & 1 \end{pmatrix} \mid a,b,c \in K \right\},$$
$$K_3 = \left\{ \begin{pmatrix} 1 & at & 0 \\ 0 & 1 & 0 \\ bt & ct^2 & 1 \end{pmatrix} \mid a,b,c \in K \right\}.$$

In particular, φ is an isomorphism from $G_H = \langle H_i, i = 1, 2, 3 \rangle$ to $G_K = \langle K_i, i = 1, 2, 3 \rangle$ and hence also from $\mathcal{CC}(G_H; (H_i)_{i=1}^3)$ to $\mathcal{CC}(G_K; (K_i)_{i=1}^3)$.

To summarize, we observe that the KMS complexes are not a direct generalization of the previous constructions, but it seems that the underlying structure which implies good expansion is the same.

Chapter 5

Cone functions

Arguably the only known way for showing coboundary expansion is via the cone method. The idea was first used by Gromov in [Gro10] and was then further developed in [LMM16; KO21; KM19; DD24a; DD24c].

The motivation for cone functions comes from the following result for very symmetric graphs.

Theorem 5.0.1 ([Chu97, Theorem 7.1])**.** Let X be a finite connected graph such that its automorphism group acts transitively on the edges. Let D denote its diameter, i.e. $D = \max_{u,v \in X(0)} \min\{\text{length}(P) \mid P \text{ path between } u \text{ and } v\}$ and let $h(X)$ be its Cheeger constant. Then $h(X) \geq \frac{1}{2D}$.

The idea of cone functions is to generalize this result to higher dimensions and the coboundary expansion.

In this chapter, we introduce cone functions, both in the case of non-abelian and abelian coefficients and describe some technical tools for constructing cone functions. These results have been published in [OV25a; OV25b].

5.1 Non-abelian cone functions

Non-abelian cone functions were introduced in [DD24c], our definition is very similar. We follow the exposition in [OV25b].

Throughout, X is an n-dimensional simplicial complex. We again denote by $X_{\text{ord}}(k)$ the set of ordered k-dimensional simplices of X, and for a simplex $\{v_0, \ldots, v_k\} \in X$ we denote by $(v_0, \ldots, v_k) \in X_{\text{ord}}$ the ordered simplex.

Given two vertices u, v of X, a *path from u to v* is a $(k+1)$-tuple of vertices $(u_0; ...; u_k)$ where $u = u_0$ and $v = u_k$ such that for every $0 \leq i \leq k-1$, $(u_i, u_{i+1}) \in X_{\text{ord}}(1)$. The constant k is called the *length of the path.* Note that we use ; to differentiate between paths and ordered simplices. Given two paths $P_0 = (u_0; ...; u_k), P_1 = (u_k; ...; u_m)$, we define $P_0 \circ P_1 = (u_0; ...; u_m)$. Also, given a path $P = (u_0; ...; u_k)$, we denote $P^{-1} = (u_k; ...; u_0)$.

A *loop around v* is a path from v to itself. We note that (v) is also considers as a loop of length 0 and we will call it the *trivial loop around v*.

Definition 5.1.1. Fix v_0 to be a vertex in X. We define the following symmetric relations between two loops P, P' around v_0, see also their visualizations in Figure 5.1.

- We denote $P \overset{(BT)}{\sim} P'$ if there exist vertices u, v and paths Q_0, Q_1 from v_0 to u such that $P = Q_0 \circ (u; v; u) \circ Q_1^{-1}$ and $P' = Q_0 \circ (u) \circ Q_1^{-1}$ ((BT) stands for "backtracking").
- We denote $P \overset{(TR)}{\sim} P'$ if there is a triangle (u, v, w) in X and paths Q_0, Q_1 from v_0 to u and w correspondingly such that $P = Q_0 \circ (u; v; w) \circ Q_1^{-1}$ and $P' = Q_0 \circ (u; w) \circ Q_1^{-1}$ ((TR) stands for "triangle"). We denote by $t_{P,P'} = (u, v, w)$ the triangle used for the relation.

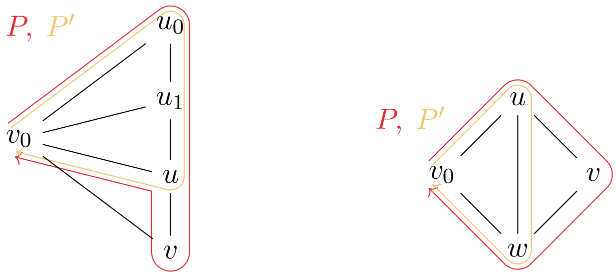


Figure 5.1: Visualization of $\overset{(BT)}{\sim}$ on the left and $\overset{(TR)}{\sim}$ on the right.

Remark 5.1.2. Note that the relations above are not defined to be transitive. Indeed, in the sequel, it will be important to keep track of how many times the relation $\overset{(TR)}{\sim}$ is used to pass from one loop to another.

Remark 5.1.3. Let P, P' be two loops around a vertex v in X. If $P \overset{(BT)}{\sim} P'$ or $P \overset{(TR)}{\sim} P'$ then the corresponding paths in the geometric realization of X are homotopic. Conversely, any two loops P, P' in $|X|$ along the edges of X around v which are homotopic can be seen as loops in the abstract simplicial complex and we can find a sequence of loops $P = P_0, \ldots, P_m = P'$ such that $P_i \overset{(BT)}{\sim} P_{i+1}$ or $P_i \overset{(TR)}{\sim} P_{i+1}$ for all $0 \leq i \leq m-1$.

Definition 5.1.4. A 0-*cone* on X is a pair $\mathrm{Cone}_X = \left(v_0, (P_u)_{u \in X(0)}\right)$ such that

(i) $v_0 \in X(0)$.

(ii) For every $u \in X(0), u \neq v_0$, P_u is a path from v_0 to u. For v_0, P_{v_0} is the trivial loop around v_0.

A 1-*cone* on X is a triple $\mathrm{Cone}_X = \left(v_0, (P_u)_{u \in X(0)}, (T_{(u,v)})_{(u,v) \in X_{\mathrm{ord}}(1)}\right)$ such that

(i) $\left(v_0, (P_u)_{u \in X(0)}\right)$ is a 0-cone on X.

(ii) For every $(u, v) \in X_{\mathrm{ord}}(1)$, $T_{(u,v)}$ is a sequence of loops around v_0, $T_{(u,v)} = (P_0, P_1, \ldots, P_m)$ such that the following holds:

(i) $P_0 = P_u \circ (u;v) \circ P_v^{-1}$.

(ii) For every $0 \leq i \leq m-1$, either $P_i \overset{(TR)}{\sim} P_{i+1}$ or $P_i \overset{(BT)}{\sim} P_{i+1}$.

(iii) P_m is the trivial loop around v_0.

We refer to the fixed vertex v_0 as the apex of the cone. Furthermore, we call $T_{(u,v)} = (P_0, P_1, ..., P_m)$ as above a *contraction* and denote the number of triangle relations in the contraction by

$$|T_{(u,v)}| = |\{0 \leq i \leq m-1 : P_i \overset{(TR)}{\sim} P_{i+1}\}|$$

and the set of triangles used in the construction by

$$\mathcal{T}_{(u,v)} = \{t_{P_\ell, P_{\ell+1}} \mid P_\ell \overset{(TR)}{\sim} P_{\ell+1} \in T_{(u,v)}\}.$$

We further define the *radii of the cone* as follows:

(i) $\mathrm{Rad}_0(\mathrm{Cone}_X) = \sup_{u \in X(0)} \mathrm{length}(P_u)$.

(ii) $\mathrm{Rad}_1(\mathrm{Cone}_X) = \sup_{(u,v) \in X_{\mathrm{ord}}(1)} |T_{(u,v)}|$.

(iii) $\mathrm{Rad}(\mathrm{Cone}_X) = \begin{cases} \max\{\mathrm{Rad}_0(\mathrm{Cone}_X), \mathrm{Rad}_1(\mathrm{Cone}_X)\} & \text{if } \mathrm{Cone}_X \text{ is a 1-cone} \\ \mathrm{Rad}_0(\mathrm{Cone}_X) & \text{if } \mathrm{Cone}_X \text{ is a 0-cone} \end{cases}$.

Remark 5.1.5. The definition of $|T_{(u,v)}|$ in [DD24c] is slightly different than ours. Namely, they define another relation $\sim_1$ that is composed of applying $\overset{(TR)}{\sim}$ once and $\overset{(BT)}{\sim}$ as many times as needed. Then they define $|T_{(u,v)}|$ as how many such relations $\sim_1$ are needed to bring P_0 to P_m such that P_m can be made trivial by only applying the relations $\overset{(BT)}{\sim}$. But this definition results in the same cone radius.

Remark 5.1.6. Any connected non-empty simplicial complex admits a 0-cone: pick an arbitrary vertex v_0, and by connectivity, we can find a path from any other vertex u to v_0. The converse is also true.

Similarly, any 1-connected (connected and simply connected) simplicial complex X admits a 1-cone function. By connectedness we get a 0-cone. Then each loop $P_u \circ (u;v) \circ P_v^{-1}$ is contractible in the geometric realization of X. By Remark 5.1.3, we can realize this homotopy equivalence by a sequence of paths connected by the relations $\overset{(BT)}{\sim}, \overset{(TR)}{\sim}$. The converse is also true, as we will see in the following lemma.

Lemma 5.1.7. Let X be a finite n-dimensional simplicial complex with 1-cone $\mathrm{Cone}_X = \left(v_0, (P_u)_{u \in X(0)}, \left(T_{(u,v)}\right)_{(u,v) \in X_{\mathrm{ord}}(1)}\right)$. Then $|X|$ is 1-connected.

Proof. Note that X is connected and hence $|X|$ is path connected. Hence we can without loss of generality consider a loop around v_0 in $|X|$. This loop is homotopic to a loop along the edges of X. Let $P = (v_0; u_1; \dots; u_m; v_0)$ denote this loop. Then

$$\begin{aligned} P =& (v_0) \circ (v_0; u_1; \dots; u_m; v_0) \sim_{T_{(u_1,v_0)}} P_{u_1} \circ (u_1; v_0) \circ (v_0; u_1; \dots; u_m; v_0) \\ &\overset{(BT)}{\sim} P_{u_1} \circ (u_1; \dots; u_m; v_0) \overset{(BT)}{\sim} \dots \overset{(BT)}{\sim} P_{u_1} \circ (u_1; u_2) \circ P_{u_2}^{-1} \circ P_{u_2} \circ (u_2; \dots; u_m; v_0) \\ &\sim_{T_{(u_1,u_2)}} P_{u_2} \circ (u_2; \dots; u_m; v_0). \end{aligned}$$

Here we denoted by $\sim_{T_{(u_i,u_{i+1})}}$ the series of relations that are described by $T_{(u_i,u_{i+1})}$.

Further iterating this process yields

$$P \sim P_{u_m} \circ (u_m; v_0) \sim_{T_{(u_m,v_0)}} (v_0).$$

Thus we have shown that P is contractible and hence $|X|$ is simply connected. □

A bound on the cone radius gives rise to a bound on the coboundary expansion via the following proposition. Thus we can intuitively think of coboundary expansion as a quantitative way of (higher) connectedness. This result was first shown in [DD24c, Lemma 1.6, Lemma 4.3] in a slightly more general context. We provide a proof for completeness, adapted to our setting.

Proposition 5.1.8. Let X be a pure n-dimensional finite simplicial complex, $n \geq 2$. Assume that there is a 1-cone on X such that $\mathrm{Rad}(\mathrm{Cone}_X) < \infty$ and that the automorphism group $\mathrm{Aut}(X)$ acts transitively on $X(n)$. Then for every group Λ,

$$h^1_{\mathrm{cb}}(X, \Lambda) \geq \frac{1}{\binom{n+1}{3} \mathrm{Rad}_1(\mathrm{Cone}_X)}.$$

We split the proof into smaller pieces.

The group $G = \mathrm{Aut}(X)$ acts on X simplicially. This induces an action on X_{ord} via $g.(v_0, \dots, v_k) = (g.v_0, \dots, g.v_k)$ and on paths by similarly acting on each vertex of the path. We set

$$g.C = \left(g.v_0, (g.P_{g^{-1}.u})_{u \in X(0)}, (g.T_{(g^{-1}u, g^{-1}v)})_{(u,v) \in X_{\mathrm{ord}}(1)}\right).$$

Lemma 5.1.9. $g.C$ is a 1-cone with apex $g.v_0$.

Proof. Let $u \in X(0)$, $g \in G$ and set $u' = g^{-1}.u$. Let $P_{u'} = (v_0; u_1; \dots, u_m; u')$. Then $g.P_{u'} = (g.v_0; g.u_1; \dots; g.u_m; g.u')$ is still a path (since the action of G is simplicial and hence preserves edges) now with start $g.v_0$ and end at $g.u' = (gg^{-1}).u = u$.

Furthermore, note that the action of G preserves the relations $\overset{(BT)}{\sim}$, $\overset{(TR)}{\sim}$. Thus $g.C$ is a 1-cone with apex $g.v_0$ and cone radius equal to the cone radius of C. □

For $\tau \in X_{\mathrm{ord}}$ denote by $\tilde{\tau} \in X$ the unordered simplex on the same set of vertices. Recall the we so far defined the Garland weight w on the unordered simplices of X. We define the weight of an ordered simplex $\tau \in X_{\mathrm{ord}}(k)$ as $w(\tau) = \frac{1}{(k+1)!} w(\tilde{\tau})$. Thus we still have that $\sum_{\tau \in X_{\mathrm{ord}}(k)} w(\tau) = 1$ for all $-1 \leq k \leq n$. For a set of oriented triangles from a 1-cone $\mathcal{T}_{(u,v)}$ we write $\widetilde{\mathcal{T}_{(u,v)}} = \{\tilde{\tau} \mid \tau \in \mathcal{T}_{(u,v)}\}$ for the set containing the unoriented versions of the triangles.

Lemma 5.1.10. Assume that G acts transitively on $X(n)$. For $\tilde{\sigma} \in X(2)$ define

$$\theta(\tilde{\sigma}) = \frac{1}{w(\tilde{\sigma})|G|} \sum_{g \in G} \sum \{ w((u,v)) \mid (u,v) \in X_{\mathrm{ord}}(1), \tilde{\sigma} \in g.\tilde{\mathcal{T}}_{(g^{-1}.u, g^{-1}.v)} \}.$$

Then

$$\theta(\tilde{\sigma}) \leq \binom{n+1}{3} \mathrm{Rad}_1(C).$$

Here and below, we mean by $\sum \{ w(\tau) \mid \tau \in A \}$ for some set A that we take the sum $\sum_{\tau \in A} w(\tau)$. But this notation is more convenient when the description of A is very long.

Proof. Fix a $\tilde{\sigma} \in X(2)$. Note that every $g \in G_{\tilde{\sigma}} = \mathrm{Stab}_G(\tilde{\sigma})$ stabilizes the set $\{ \tilde{\tau} \in X(n) \mid \tilde{\sigma} \subseteq \tilde{\tau} \}$. Thus we get the following

$$\bigcup_{g \in G/G_{\tilde{\sigma}}} g.\{ \tilde{\tau} \in X(n) \mid \tilde{\sigma} \subseteq \tilde{\tau} \} = X(n)$$

where the equality holds by transitivity of the action on $X(n)$. Thus we get

$$|X(n)| \leq \frac{|G|}{|G_{\tilde{\sigma}}|} |\{ \tilde{\tau} \in X(n) \mid \tilde{\sigma} \subseteq \tilde{\tau} \}|$$

and hence

$$|G_{\tilde{\sigma}}| \leq |G| \binom{n+1}{3} w(\tilde{\sigma})$$

where we use that

$$w(\tilde{\sigma}) = \frac{|\{ \tilde{\tau} \in X(n) \mid \tilde{\sigma} \subseteq \tilde{\tau} \}|}{\binom{n+1}{3} |X(n)|}.$$

Note that, since the action of G is simplicial, w is G-invariant. Thus we have

$$\begin{aligned}
\theta(\tilde{\sigma}) &= \frac{1}{w(\tilde{\sigma})|G|} \sum_{g \in G} \sum \{ w((u,v)) \mid (u,v) \in X_{\mathrm{ord}}(1), \tilde{\sigma} \in g.\tilde{\mathcal{T}}_{(g^{-1}.u, g^{-1}.v)} \} \\
&= \frac{1}{w(\tilde{\sigma})|G|} \sum_{g \in G} \sum \{ w(g^{-1}.(u,v)) \mid (u,v) \in X_{\mathrm{ord}}(1), g^{-1}.\tilde{\sigma} \in \tilde{\mathcal{T}}_{(g^{-1}.u, g^{-1}.v)} \} \\
&= \frac{1}{w(\tilde{\sigma})|G|} \sum_{g \in G} \sum \{ w((u,v)) \mid (u,v) \in X_{\mathrm{ord}}(1), g^{-1}.\tilde{\sigma} \in \tilde{\mathcal{T}}_{(u,v)} \} \\
&= \frac{1}{w(\tilde{\sigma})|G|} \sum_{(u,v) \in X_{\mathrm{ord}}(1)} w((u,v)) |\{ g \in G \mid g^{-1}.\tilde{\sigma} \in \tilde{\mathcal{T}}_{(u,v)} \}| \\
&\leq \frac{1}{w(\tilde{\sigma})|G|} \sum_{(u,v) \in X_{\mathrm{ord}}(1)} w((u,v)) |\tilde{\mathcal{T}}_{(u,v)}| \cdot |G_{\tilde{\sigma}}| \\
&\leq \frac{1}{w(\tilde{\sigma})} \mathrm{Rad}_1(C) \frac{|G_{\tilde{\sigma}}|}{|G|} \sum_{(u,v) \in X_{\mathrm{ord}}(1)} w((u,v)) \\
&\leq \binom{n+1}{3} \mathrm{Rad}_1(C) \sum_{\tau \in X_{\mathrm{ord}}(1)} w(\tau) = \binom{n+1}{3} \mathrm{Rad}_1(C).
\end{aligned}$$

□

Definition 5.1.11. Let X be a pure n-dimensional, finite simplicial complex, $n \geq 2$. Assume that $G = \mathrm{Aut}(X)$ acts transitively on $X(n)$. Let Γ be an arbitrary coefficient group. Given a cochain $f \in C^1(X, \Gamma)$ and a path $P = (u_0; \dots; u_m)$ we define

$$f(P) = f((u_{m-1}, u_m)) \cdot \ldots \cdot f((u_1, u_2)) \cdot f((u_0, u_1)) \in \Gamma.$$

We further define that f applied to a trivial loop is $f((v)) = 1$. Furthermore, given a 1-cone of X, $C = \mathrm{Cone}_X = \big(v_0, (P_u)_{u \in X(0)}, (T_{(u,v)})_{(u,v) \in X_{\mathrm{ord}}(1)}\big)$ with bounded radius and $g \in G$, we define the map $S_g : C^1(X, \Gamma) \to C^0(X, \Gamma)$ by setting $S_g f(u) = f(g.P_{g^{-1}.u})$ for all $f \in C^1(X, \Gamma), u \in X(0)$.

Lemma 5.1.12. Let $P \overset{(TR)}{\sim} P'$. If $(d_1 f)(t_{P,P'}) = 1$ then $f(P) = f(P')$. Furthermore, if $P \overset{(BT)}{\sim} P'$ then $f(P) = f(P')$.

Proof. First of all, note that for $f \in C^1(X, \Gamma)$ and paths $P = (u_0; \dots; u_m), Q = (u_m; \dots; u_{m+\ell})$, we have

$$f(P \circ Q) = f((u_{m+\ell-1}, u_{m+\ell})) \dots f(u_m, u_{m+1}) f((u_{m-1}, u_m)) \dots f((u_0, u_1)) = f(Q)f(P).$$

We start with the case $P \overset{(BT)}{\sim} P'$. Thus we can write $P = Q_0 \circ (u; v; u) \circ Q_1^{-1}$ and $P' = Q_0 \circ (u) \circ Q_1^{-1}$. Hence for $f \in C^1(X, \Gamma)$ we have, using that for cochains $f((u, v)) = f((v, u))^{-1}$,

$$f(P) = f(Q_1^{-1}) f((v, u)) f((u, v)) f(Q_0) = f(Q_1^{-1}) f(Q_0) = f(P').$$

Next, assume $P \overset{(TR)}{\sim} P'$ and hence we can write $P = Q_0 \circ (u; v; w) \circ Q_1^{-1}, P' = Q_0 \circ (u; w) \circ Q_1^{-1}$. Note that $df((u, v, w)) = f((u, v)) f((w, u)) f((v, w))$. If $df((u, v, w)) = 1$ then $f((u, w)) = f((v, w)) f((u, v))$. Together we get

$$\begin{aligned} f(P) &= f(Q_1^{-1}) f((v, w)) f((u, v)) f(Q_0) \\ &= f(Q_1^{-1}) f((u, w)) f(Q_0) = f(P'). \end{aligned}$$

$\square$

Lemma 5.1.13. Let $g \in G$, $(u, v) \in X_{\mathrm{ord}}(1)$ and $g.T_{(g^{-1}.u, g^{-1}.v)} = (P_0, \dots, P_m)$. If $f(P_i) = f(P_{i+1})$ for all $0 \leq i \leq m - 1$ then $f((u, v)) = (d_0 S_g f)((u, v))$.

Proof. First of all, note that we have

$$\begin{aligned} f((u_0; \dots; u_m)^{-1}) &= f((u_m; \dots; u_0)) = f((u_1, u_0)) f((u_2, u_1)) \dots f((u_m, u_{m-1})) \\ &= f((u_0; \dots; u_m))^{-1}. \end{aligned}$$

Using the assumption we get

$$\begin{aligned} &1 = f(P_m) = f(P_0) = f(g.P_{g^{-1}u} \circ (u; v) \circ g.P^{-1}_{g^{-1}.v}) = f(g.P_{g^{-1}.v})^{-1} f((u, v)) f(g.P_{g^{-1}u}) \\ &\iff f(g.P_{g^{-1}.v}) f(g.P_{g^{-1}u})^{-1} = f((u, v)). \end{aligned}$$

Thus

$$(d_0 S_g f)((u,v)) = S_g f(v) S_g f(u)^{-1} = f(g.P_{g^{-1}.v}) f(g.P_{g^{-1}.u})^{-1} = f((u,v)).$$

□

Proof of Proposition 5.1.8. By Lemmas 5.1.12 and 5.1.13 we have that if $d_1 f(g.t) = 1$ for all $t \in \mathcal{T}_{(g^{-1}.u, g^{-1}.v)}$ then $f((u,v)) = d_0 S_g f((u,v))$. Thus

$$\begin{aligned}
\operatorname{dist}(f, d_0 S_g f) &= \sum \{ w((u,v)) \mid (u,v) \in X_{\mathrm{ord}}(1) : f((u,v)) \neq (d_0 S_g f)((u,v)) \} \\
&\leq \sum \{ w((u,v)) \mid (u,v) \in X_{\mathrm{ord}}(1) : \exists t \in \mathcal{T}_{(g^{-1}.u, g^{-1}.v)} : df(g.t) \neq 1 \} \\
&= \sum \{ w((u,v)) \mid (u,v) \in X_{\mathrm{ord}}(1) : g.\tilde{\mathcal{T}}_{(g^{-1}.u, g^{-1}.v)} \cap \operatorname{supp}(df) \neq \emptyset \}.
\end{aligned}$$

Finally, we bound

$$h^1_{\mathrm{cb}}(X, \Lambda) = \min_{f \in C^1(X,\Lambda) \setminus B^1(X,\Lambda)} \frac{\|d_1 f\|}{\min_{\psi \in B^1(X,\Lambda)} \operatorname{dist}(f, \psi)}.$$

Fix $f \in C^1(X, \Gamma)$. Observe that $\min_{\psi \in B^1(X,\Lambda)} \operatorname{dist}(f,\psi) \leq \operatorname{dist}(f, d_0 S_g f)$ for all $g \in G$. Averaging over all $g \in G$ we get

$$\begin{aligned}
\min_{\psi \in B^1(X,\Lambda)} \operatorname{dist}(f,\psi) &\leq \frac{1}{|G|} \sum_{g \in G} \operatorname{dist}(f, d_0 S_g f) \\
&\leq \frac{1}{|G|} \sum_{g \in G} \sum \{ w((u,v)) \mid (u,v) \in X_{\mathrm{ord}}(1) : g.\tilde{\mathcal{T}}_{(g^{-1}.u, g^{-1}.v)} \cap \operatorname{supp}(df) \neq \emptyset \} \\
&\leq \frac{1}{|G|} \sum_{\tilde{\sigma} \in \operatorname{supp}(df)} \sum_{g \in G} \sum \{ w((u,v)) \mid (u,v) \in X_{\mathrm{ord}}(1) : \tilde{\sigma} \in g.\tilde{\mathcal{T}}_{(g^{-1}.u, g^{-1}.v)} \} \\
&= \sum_{\tilde{\sigma} \in \operatorname{supp}(df)} \frac{w(\tilde{\sigma})}{w(\tilde{\sigma})|G|} \sum_{g \in G} \sum \{ w((u,v)) \mid (u,v) \in X_{\mathrm{ord}}(1) : \tilde{\sigma} \in g.\mathcal{T}_{(g^{-1}.u, g^{-1}.v)} \} \\
&= \sum_{\tilde{\sigma} \in \operatorname{supp}(df)} w(\tilde{\sigma}) \theta(\tilde{\sigma}) \\
&\leq \left(\sum_{\tilde{\sigma} \in \operatorname{supp}(df)} w(\tilde{\sigma}) \right) \binom{n+1}{3} \operatorname{Rad}_1(C) = \|df\| \binom{n+1}{3} \operatorname{Rad}_1(C).
\end{aligned}$$

Thus, we get as needed

$$\frac{\|df\|}{\min_{\psi \in B^1(X,\Lambda)} \operatorname{dist}(f,\psi)} \geq \frac{1}{\binom{n+1}{3} \operatorname{Rad}_1(C)}.$$

Remark 5.1.14. We can get a similar result as above if G acts transitively on $X(k)$ for some $2 \leq k \leq n$. In that case, we can either consider the Garland weights with respect to the k-skeleton, then we are back in the situation above. If we want to keep considering the

Garland weights taking the n-dimensional faces into account, we can adjust the procedure as follows. Let $d = \max_{\tau \in X(k)} |\{\sigma \in X(n) \mid \tau \subseteq \sigma\}|$. Note that in the setting of infinite families of HDX this will always be uniformly bounded. We then have to adapt the proof as follows. We have $|G_{\tilde{\sigma}}| \leq |G| \binom{k+1}{3} w(\tilde{\sigma}) d$ and thus $\theta(\tilde{\sigma}) \leq \binom{k+1}{3} \operatorname{Rad}_1(C) d$ and hence $h^1_{\mathrm{cb}}(X, \Gamma) \geq \frac{1}{d \binom{k+1}{3} \operatorname{Rad}_1(C)}$.

5.2 Abelian cone functions

We have so far developed a tool for showing 1-coboundary expansion for any coefficient group. We now want to extend this theory to show coboundary expansion for higher dimensions, now of course with respect to an abelian group as coefficients.

We fix a finite, n-dimensional simplicial complex X and an abelian group $\mathbb{A}$. Recall that for every oriented simplex $\sigma \in \overrightarrow{X}(k), a \in \mathbb{A}$ we defined the chain $a\mathbb{1}_\sigma = \sum_{\tau \in \overrightarrow{X}(k)} t_\tau \tau$ such that $t_\sigma = a, t_{-\sigma} = -a$ and $t_\tau = 0$ in all other cases. Then every element in $C_k(X, \mathbb{A})$ can be written as a sum of $a\mathbb{1}_\sigma$'s.

Definition 5.2.1. Let X be a finite simplicial complex. Let $k \in \mathbb{N} \cup \{0, -1\}$ be a constant, $\mathbb{A}$ an abelian group and v be a vertex of X. A $(k, \mathbb{A})$-cone function with apex v is a collection of functions $\operatorname{Cone}_X^j : C_j(X, \mathbb{A}) \to C_{j+1}(X, \mathbb{A})$ for $-1 \leq j \leq k$ that fulfills the following conditions:

(i) For every $a \in \mathbb{A}$, $\operatorname{Cone}_X^{-1}(a\mathbb{1}_\emptyset) = a\mathbb{1}_{[v]}$.

(ii) For every $-1 \leq j \leq k$, Cone_X^j is a homomorphism of $\mathbb{Z}$-modules.

(iii) For every $0 \leq j \leq k$ and every $A \in C_j(X, \mathbb{A})$,

$$\partial_{j+1} \operatorname{Cone}_X^j(A) + \operatorname{Cone}_X^{j-1}(\partial_j A) = A.$$

We will write $\operatorname{Cone}_X : \bigoplus_{j=-1}^{k} C_j(X, \mathbb{A}) \to \bigoplus_{j=0}^{k+1} C_j(X, \mathbb{A})$ combining all the Cone_X^j. Sometimes, for shorter notation, we will follow the convention of Wild in [Wil22] and write S_j for Cone_X^j.

Remark 5.2.2. Below, we will refer to

$$\partial_{j+1} \operatorname{Cone}_X(A) + \operatorname{Cone}_X(\partial_j A) = A$$

as *the cone equation.* It imposes that Cone_X is a chain homotopy between the identity and the zero-map on $C_j(X, \mathbb{A}), -1 \leq j \leq k$. Intuitively, is makes sense to consider this, since a topological space is contractible if and only if there is a homotopy between the identity map on the space and a map that is constantly one point of the space.

Lemma 5.2.3. Let $(v_0, (P_u)_{u \in X(0)}, (T_{(u,v)})_{(u,v) \in X_{\mathrm{ord}}(1)})$ be a 1-cone in the sense of Definition 5.1.4. Then it gives rise to a $(1, \mathbb{A})$-cone function for every abelian group $\mathbb{A}$. This also holds for both types of 0-cone functions.

Proof. Let $(v_0, (P_u)_{u\in X(0)}, (T_{(u,v)})_{(u,v)\in X_{\mathrm{ord}}(1)})$ be a 1-cone and $\mathbb{A}$ an abelian group. For each $a \in \mathbb{A}$, we define the map ψ_a from the set of all paths in X to $C_1(X, \mathbb{A})$ as

$$\psi_a((u_0; \ldots ; u_m)) = \sum_{i=0}^{m-1} a\mathbb{1}_{[u_i,u_{i+1}]}.$$

First of all, note that $\psi_a(P \circ Q) = \psi_a(P) + \psi_a(Q)$.

Claim 1: If $P \overset{(BT)}{\sim} P'$ then $\psi_a(P) = \psi_a(P')$.

To show the claim, assume $P \overset{(BT)}{\sim} P'$ are loops around v_0 in X. Thus we can write $P = Q_0 \circ (u; v; u) \circ Q_1^{-1}, P' = Q_0 \circ (u) \circ Q_1^{-1}$. Hence we have

$$\begin{aligned}\psi_a(P) &= \psi(Q_0) + \psi_a((u; v; u) + \psi_a(Q_1^{-1}))\\ &= \psi_a(Q_0) + a\mathbb{1}_{[u,v]} + a\mathbb{1}_{[v,u]} + \psi_a(Q_1^{-1})\\ &= \psi_a(Q_0) + a\mathbb{1}_{[u,v]} - a\mathbb{1}_{[u,v]} + \psi_a(Q_1^{-1})\\ &= \psi_a(Q_0) + \psi_a(Q_1^{-1}) = \psi_a(P').\end{aligned}$$

Claim 2: If $P \overset{(TR)}{\sim} P'$ via triangle (u, v, w) then $\psi_a(P) = \psi_a(P') + \partial a\mathbb{1}_{[u,v,w]}$.

Since $P \overset{(TR)}{\sim} P'$ we can write $P = Q_0 \circ (u; v; w) \circ Q_1^{-1}, P' = Q_0 \circ (u; w) \circ Q_1^{-1}$. Thus

$$\psi_a(P) - \psi_a(P') = \psi_a((u; v; w)) - \psi_a((u; w)) = a\mathbb{1}_{[u,v]} + a\mathbb{1}_{[v,w]} - a\mathbb{1}_{[u,w]} = \partial a\mathbb{1}_{[u,v,w]}.$$

Now to define the $(1, \mathbb{A})$-cone function, we set

$$\begin{aligned}S_{-1}(a\mathbb{1}_{\emptyset}) &= a\mathbb{1}_{[v_0]}\\ S_0(a\mathbb{1}_{[v_0]}) &= 0;\ S_0(a\mathbb{1}_{[u]}) = \psi_a(P_u)\\ S_1(a\mathbb{1}_{[u,w]}) &= \sum_{t\in\mathcal{T}_{(u,w)}} a\mathbb{1}_t\end{aligned}$$

where $\mathcal{T}_{(u,w)}$ is the set of triangles used for $\overset{(TR)}{\sim}$ in the contraction $T_{(u,w)}$. We check the cone equation, first for $w \in X(0), w \neq v_0, P_w = (w_0, \ldots , w_m)$ with $w_0 = v$ and $w_m = w$.

$$\partial S_0(a\mathbb{1}_{[w]}) + S_{-1}(\partial a\mathbb{1}_{[w]}) = \partial\left(\sum_{i=0}^{m-1} a\mathbb{1}_{[w_i,w_{i+1}]}\right) + a\mathbb{1}_{[v_0]} = a\mathbb{1}_{[w]} - a\mathbb{1}_{[v_0]} + a\mathbb{1}_{[v]} = a\mathbb{1}_{[w]}.$$

Now; for $[u, w] \in \overrightarrow{X}(1)$ we have

$$\begin{aligned}\partial S_1(a\mathbb{1}_{[u,w]}) + S_0(\partial a\mathbb{1}_{[u,w]}) &= \sum_{t\in\mathcal{T}_{(u,w)}} \partial a\mathbb{1}_t + \psi_a(P_w) - \psi_a(P_u)\\ &= -\psi_a(P_u \circ (u; w) \circ P_w^{-1}) + \sum_{t\in\mathcal{T}_{(u,w)}} \partial a\mathbb{1}_t + a\mathbb{1}_{[u,w]}\\ &\overset{(*)}{=} \psi_a((v_0)) + a\mathbb{1}_{[u,w]} = a\mathbb{1}_{[u,w]}\end{aligned}$$

where the equality $(*)$ holds by iteratively applying Claim 1 and 2 on the contraction $T_{(u,w)}$. □

The following result shows how the existence of a cone function is connected to the vanishing of homology. A dual version of this is shown in [Wil22].

Lemma 5.2.4. Let X be a finite, n-dimensional simplicial complex. If there exists a $(k, \mathbb{A})$-cone function of X then $H_j(X, \mathbb{A}) = 0$ for all $0 \leq j \leq k$.

If $\mathbb{A}$ is a commutative unital ring then the converse is also true.

Proof. We start by assuming that we have a $(k, \mathbb{A})$-cone function S. Let $c \in C_i(X, \mathbb{A}), 0 \leq i \leq k$ such that $\partial c = 0$. Then $c = \partial S_i(c) + S_{i-1}(\partial c) = \partial S_i(c)$ and hence $c \in \mathrm{Im}(\partial_{i+1})$. Thus $Z_i(X, \mathbb{A}) = B_i(X, \mathbb{A})$ and hence $H_i(X, \mathbb{A}) = 0$.

For the other direction, assume that $\mathbb{A}$ is a commutative unital ring and that $H_j(X, \mathbb{A}) = 0$ for all $0 \leq j \leq k$. We construct a cone function from here. Fix an arbitrary vertex $v_0 \in X$. We set

$$S_{-1}(\mathbb{1}_\emptyset) = \mathbb{1}_{[v_0]} \text{ and } S_0(\mathbb{1}_{[v_0]}) = 0.$$

Let $w \in X(0), w \neq v_0$. We want to define $S_0(\mathbb{1}_{[w]})$ such that it satisfies the cone equation, i.e. $\mathbb{1}_{[w]} = \partial S_0(\mathbb{1}_{[w]}) + S_{-1}(\partial \mathbb{1}_{[w]})$. This yields the condition $\partial S_0(\mathbb{1}_{[w]}) = \mathbb{1}_{[w]} - \mathbb{1}_{[v]}$. Next, observe that $\partial(\mathbb{1}_{[w]} - \mathbb{1}_{[v_0]}) = \mathbb{1}_\emptyset - \mathbb{1}_\emptyset = 0$ and hence $\mathbb{1}_{[w]} - \mathbb{1}_{[v_0]} \in \ker(\partial_0) = \mathrm{Im}(\partial_1)$, where the equality holds since $H_0(X, \mathbb{A}) = 0$. Thus we can find $c \in C_1(X, \mathbb{A})$ such that $\partial c = \mathbb{1}_{[w]} - \mathbb{1}_{[v_0]}$ and we set $S_0(\mathbb{1}_{[w]}) = c$.

We proceed inductively. Assume we have defined S_j for $j \leq i$. Let $\sigma \in \overrightarrow{X}(i+1)$. We want to define $S_{i+1}(\mathbb{1}_\sigma)$. To satisfy the cone equation, we need to fullfil $\partial S_{i+1}(\mathbb{1}_\sigma) = \mathbb{1}_\sigma - S_i(\partial \mathbb{1}_\sigma)$. Now $\partial(\mathbb{1}_\sigma - S_i(\partial \mathbb{1}_\sigma)) = \partial \mathbb{1}_\sigma - \partial S_i(\partial \mathbb{1}_\sigma) = S_{i-1}(\partial(\partial \mathbb{1}_\sigma)) = 0$ using the cone equation for S_i. Thus we can find an element $c \in C_{i+2}(X, \mathbb{A})$ such that $\partial c = \mathbb{1}_\sigma - S_i(\partial \mathbb{1}_\sigma)$ and we set $S_{i+1}(\mathbb{1}_\sigma) = c$.

Thus extending the S_i $\mathbb{A}$-linearly, they define a $(k, \mathbb{A})$-cone function.

□

By the above lemma, we can see that the existence of cone functions for a simplicial complex imply higher dimensional connectivity. We want to quantify this by, similar to the non-abelian case, measure the radius of a cone. Recall that for a chain $A = \sum_{\sigma \in \overrightarrow{X}(k)} t_\sigma \sigma \in C_k(X, \mathbb{A})$ we define $\mathrm{supp}(A) = \{\{v_0, \ldots, v_k\} \in X(k) \mid t_{[v_0, \ldots, v_k]} \neq 0\}$.

Definition 5.2.5. Let X be an n-dimensional simplicial complex and $-1 \leq k \leq n-1$. Given a $(k, \mathbb{A})$-cone function Cone_X, we define for every $-1 \leq j \leq k$, the j-th radius of Cone_X to be

$$\mathrm{Rad}_j(\mathrm{Cone}_X) = \max_{\sigma \in \overrightarrow{X}(j), a \in \mathbb{A}} |\mathrm{supp}(\mathrm{Cone}_X(a \mathbb{1}_\sigma))|.$$

We furthermore set

$$\mathrm{Rad}(\mathrm{Cone}_X) = \max_{-1 \leq j \leq k} \mathrm{Rad}_j(\mathrm{Cone}_X).$$

And we define the $(k, \mathbb{A})$ cone radius of X to be

$$\mathrm{Rad}_k(X, \mathbb{A}) = \min\{\mathrm{Rad}_k(\mathrm{Cone}_X) : \mathrm{Cone}_X \text{ is a } (k, \mathbb{A})\text{-cone function}\}.$$

If no $(k, \mathbb{A})$-cone function of X exists, we define $\mathrm{Rad}_k(X, \mathbb{A}) = \infty$.

Remark 5.2.6. Note that in the case where $n = 0$ and X is just a non-empty set of vertices, it holds by definition that $\mathrm{Rad}_{-1}(X, \mathbb{A}) = 1$.

The work of Gromov [Gro10] and subsequent works [KO21; KM19] showed that for symmetric simplicial complexes, a bound on the cone radius gives rise to a bound on the coboundary expansion. In [KO21] the following was proven for $\mathbb{A} = \mathbb{F}_2$. The proof almost directly generalizes to arbitrary $\mathbb{A}$, we give it in detail for completeness. It is also similar to the proof of Proposition 5.1.8.

Theorem 5.2.7. Let $n \geq 1$, $\mathcal{R} > 0$ be constants and $\mathbb{A}$ a finitely generated abelian group. For every finite pure n-dimensional simplicial complex X, if

- the group $\mathrm{Aut}(X)$ acts transitively on $X(n)$,

and

- for some $0 \leq k \leq n-1$ it holds that $\mathrm{Rad}_k(X, \mathbb{A}) \leq \mathcal{R}$,

then $h^k_{\mathrm{cb}}(X, \mathbb{A}) \geq \frac{1}{\mathcal{R}\binom{n+1}{k+1}}$.

Similar to the non-abelian case, we split the proof into a few steps.

Lemma 5.2.8. Let G act on X simplicially. Then this induces an action of G on $C_j(X, \mathbb{A})$. Let Cone_X be a $(k, \mathbb{A})$-cone of X with apex v. Then for every $g \in G$ we define $\rho(g).\mathrm{Cone}_X$ as
$$\rho(g).\mathrm{Cone}^j_X : C_j(X, \mathbb{A}) \to C_{j+1}(X, \mathbb{A}) : A \mapsto g.\mathrm{Cone}^j_X(g^{-1}.A).$$
This defines an action of G on the set of all $(k, \mathbb{A})$-cone functions of X, in particular $\rho(g).\mathrm{Cone}_X$ is again a $(k, \mathbb{A})$-cone function. Furthermore, the action of G preserves the cone radius.

Proof. The action of G on X is given by an action of G on $X(0)$ which preserves the structure of X. This also induces an action on $\overrightarrow{X}$ by defining $g.[v_0, \dots, v_k] = [g.v_0, \dots, g.v_k]$ for any $[v_0, \dots, v_k] \in \overrightarrow{X}(k)$. We set $g.(a\mathbb{1}_\sigma) = a\mathbb{1}_{g.\sigma}$ for $a\mathbb{1}_\sigma \in C_j(X, \mathbb{A})$ and we extend this linearly to get an action of G on $C_j(X, \mathbb{A})$. We further note that

$$\partial g.(a\mathbb{1}_{[v_0,\dots,v_k]}) = \partial a\mathbb{1}_{[g.v_0,\dots,g.v_k]} = \sum_{i=0}^{k}(-1)^i a\mathbb{1}_{[g.v_0,\dots,g.v_{i-1},g.v_{i+1},\dots,g.v_k]} = g.(\partial a\mathbb{1}_{[v_0,\dots,v_k]}).$$

Extending linearly, we get $\partial g.A = g.(\partial A)$ for every $A \in C_k(X, \mathbb{A})$.

For easier notation, we denote $\mathrm{Cone}_X = S$ and $\rho(g).\mathrm{Cone}_X = S^g$. First of all, note that
$$S^g_j : C_j(X, \mathbb{A}) \to C_{j+1}(X, \mathbb{A}) : A \mapsto g.S(g^{-1}.A)$$
is a homomorphism of $\mathbb{Z}$-modules, since we defined the action of G on $\overrightarrow{X}(j)$ and then extended $\mathbb{A}$-linearly to get an action on $C_j(X, \mathbb{A})$. Furthermore, note that $S^g_{-1}(a\mathbb{1}_\emptyset) = g.S_{-1}(a\mathbb{1}_{g^{-1}.\emptyset}) = g.(a\mathbb{1}_{[v]}) = a\mathbb{1}_{[g.v]}$. The fact that S^g satisfies the cone equation follows from the fact that $g.\partial A = \partial g.A$ and that S satisfies it. $\square$

Lemma 5.2.9. Assume that G acts transitively on $X(n)$. Let $S = \mathrm{Cone}_X$ be a $(k, \mathbb{A})$-cone function of X. For $\sigma \in X(k+1)$ define

$$\theta(\sigma) = \frac{1}{w(\sigma)|G|} \sum_{g \in G} \sum \{w(\tau) \mid \tau \in \overrightarrow{X}(k), \sigma \in \mathrm{supp}(\rho(g).S_k(\tau))\}.$$

Then

$$\theta(\sigma) \leq \binom{n+1}{k+2} \mathrm{Rad}_k(S).$$

Proof. Fix a $\sigma \in X(k+1)$. Similar to the non-abelian case, we have for $G_\sigma = \mathrm{Stab}_G(\sigma)$ that

$$|G_\sigma| \leq |G| \binom{n+1}{k+2} w(\sigma)$$

this time using that

$$w(\sigma) = \frac{|\{\tau \in X(n) \mid \sigma \subseteq \tau\}|}{\binom{n+1}{(k+1)+1} |X(n)|}.$$

Note that, since the action by G is simplicial, w is G-invariant. Thus we have

$$\begin{aligned}
\theta(\sigma) &= \frac{1}{w(\sigma)|G|} \sum_{g \in G} \sum \{w(\tau) \mid \tau \in \overrightarrow{X}(k), \sigma \in \mathrm{supp}(g.S_k(g^{-1}.\tau))\} \\
&= \frac{1}{w(\sigma)|G|} \sum_{g \in G} \sum \{w(g^{-1}.\tau) \mid \tau \in \overrightarrow{X}(k), g^{-1}.\sigma \in \mathrm{supp}(S_k(g^{-1}.\tau))\} \\
&= \frac{1}{w(\sigma)|G|} \sum_{g \in G} \sum \{w(\tau) \mid \tau \in \overrightarrow{X}(k), g^{-1}.\sigma \in \mathrm{supp}(S_k(\tau))\} \\
&= \frac{1}{w(\sigma)|G|} \sum_{\tau \in \overrightarrow{X}(k)} w(\tau) |\{g \in G \mid g^{-1}.\sigma \in \mathrm{supp}(S_k(\tau))\}| \\
&\leq \frac{1}{w(\sigma)|G|} \sum_{\tau \in \overrightarrow{X}(k)} w(\tau) |\mathrm{supp}(S_k(\tau))| \cdot |G_\sigma| \\
&\leq \frac{1}{w(\sigma)} \mathrm{Rad}_k(S) \frac{|G_\sigma|}{|G|} \sum_{\tau \in \overrightarrow{X}(k)} w(\tau) \\
&= \binom{n+1}{k+2} \mathrm{Rad}_k(S).
\end{aligned}$$

□

Definition/Lemma 5.2.10. Let $\mathbb{A}$ be a finitely generated abelian group, and let X be a simplicial complex. For a cochain $f \in C^j(X, \mathbb{A})$ and a chain $A = \sum_{\tau \in \overrightarrow{X}(j)} \lambda_\tau \tau \in C_j(X, \mathbb{A})$ we define

$$f(A) = \sum_{\tau \in \overrightarrow{X}(j)} \lambda_\tau f(\tau).$$

Then $f(A)$ is a well defined element in $\mathbb{A}$.

Proof. Here, we need the assumption that $\mathbb{A}$ is finitely generated abelian. Then, by the structure theorem, $\mathbb{A}$ has a unital ring structure, hence the multiplication $\lambda_\tau f(\tau)$ is well defined. $\square$

Definition 5.2.11. Let S be a $(k,\mathbb{A})$-cone of X with apex v. Then we define its co-cone T as a family of maps

$$T_j : C^{j+1}(X,\mathbb{A}) \to C^j(X,\mathbb{A}), -1 \leq j \leq k$$

given by setting for $f \in C^{j+1}(X,\mathbb{A}), A \in C_j(X,\mathbb{A})$

$$T_j f(A) = f(S_j(A)).$$

Remark 5.2.12. Note that if S is a $(1,\mathbb{A})$-cone constructed from a non-abelian 1-cone as in Lemma 5.2.3, then T_0 in the above definition is the abelian version of what was called S_1 in Definition 5.1.11.

Lemma 5.2.13. Let S and T be as above. Then for $-1 \leq j \leq k, f \in C^j(X,\mathbb{A})$ we have

$$T_j(d_j f) + d_{j-1}T_{j-1}f = f.$$

Proof. Let $A \in C_j(X,\mathbb{A})$, then

$$\begin{aligned} T_j(d_j f)(A) &= d_j f(S_j(A)) \stackrel{(*)}{=} f(\partial S_j(A)) = f(A - S_{j-1}(\partial A)) \\ &= f(A) - f(S_{j-1}(\partial A)) = f(A) - T_{j-1}f(\partial A) \\ &\stackrel{(*)}{=} f(A) - d_{j-1}T_{j-1}f(A). \end{aligned}$$

The equalities marked with $(*)$ hold by the duality of homology and cohomology: it can be seen directly from the definitions that $df(A) = f(\partial A)$. $\square$

Proof of Theorem 5.2.7. By Lemma 5.2.13, we know that

$$\min_{b\in B^k(X,\mathbb{A})} \|f - b\| = \min_{b\in B^k(X,\mathbb{A})} \|T_j(d_j f) - b\|.$$

Unfortunately, it is not enough to use only one cone function, we have to use that the action of G, together with a $(k,\mathbb{A})$-cone S gives rise to a family of $(k,\mathbb{A})$-cones $\rho(g).S, g \in G$. For simplicity, we denote $\rho(g).S = S^g$ and the corresponding co-cone T^g. Then we have

$$\min_{b\in B^k(X,\mathbb{A})} \|f - b\| = \min_{b\in B^k(X,\mathbb{A})} \|T^g_j(d_j f) - b\| \leq \|T^g_j(d_j f)\|$$

(since the constant zero cochain is in $B^k(X,\mathbb{A})$) and in particular we can average over all $g \in G$ and obtain

$$\begin{aligned}
\min_{b\in B^j(X,\mathbb{A})}\|f-b\| &\le \frac{1}{|G|}\sum_{g\in G}\|T_j^g(d_jf)\| \\
&= \frac{1}{|G|}\sum_{g\in G}\sum\{w(\tilde{\tau}) \mid \tilde{\tau}\in \operatorname{supp}(T_j^g(d_jf))\} \\
&= \frac{1}{|G|}\sum_{g\in G}\sum\{w(\tau) \mid \tau\in \overrightarrow{X}(j), d_jf(S^g(\tau))\neq 0\} \\
&\le \frac{1}{|G|}\sum_{g\in G}\sum\{w(\tau) \mid \tau\in \overrightarrow{X}(j), \operatorname{supp}(d_jf)\cap\operatorname{supp}(S^g(\tau))\neq \emptyset\} \\
&= \frac{1}{|G|}\sum_{\eta\in\operatorname{supp}(d_jf)}\sum_{g\in G}\sum\{w(\tau) \mid \tau\in \overrightarrow{X}(j), \eta\in\operatorname{supp}(S^g(\tau))\} \\
&= \sum_{\eta\in\operatorname{supp}(d_jf)} w(\eta)\theta(\eta) \\
&\le \binom{n+1}{j+2}\operatorname{Rad}_j(S)\sum_{\eta\in\operatorname{supp}(d_jf)} w(\eta) = \binom{n+1}{j+2}\operatorname{Rad}_j(S)\|d_jf\|. \qquad \square
\end{aligned}$$

We will show that a bound on the cone radii of $\mathbb{Z}$-cones bounds the cone radii for every finitely generated abelian group. This result is probably well known to experts, but we could not find a reference for it, so we include a sketch of the proof here.

Proposition 5.2.14. Let X be an n-dimensional simplicial complex and $-1 \le k \le n-1$. For every finitely generated abelian group $\mathbb{A}$, it holds that $\operatorname{Rad}_k(X,\mathbb{A}) \le \operatorname{Rad}_k(X,\mathbb{Z})$.

Proof. If $(k,\mathbb{Z})$-cone functions of X do not exist, then $\operatorname{Rad}_k(X,\mathbb{Z}) = \infty$ and the inequality holds trivially. Assume that there exists a $(k,\mathbb{Z})$-cone function.

By the fundamental theorem of finitely generated abelian groups, there are prime powers $q_1,\dots,q_l$ and an integer $m \in \mathbb{N}\cup\{0\}$ such that $\mathbb{A} = \left(\bigoplus_{j=1}^l \mathbb{Z}/(q_j\mathbb{Z})\right)\oplus\mathbb{Z}^m$. We will denote the elements of $\mathbb{A}$ as $m+l$ tuples $(a_1,\dots,a_l,a_{l+1},\dots,a_{l+m})$, where for every $1\le j\le l$, $a_j \in \mathbb{Z}/(q_j\mathbb{Z})$ and $a_{l+1},\dots,a_{l+m}\in\mathbb{Z}$. We note that for every $-1\le i\le n$,

$$C_i(X,\mathbb{A}) = \left(\bigoplus_{j=1}^{l} C_i(X,\mathbb{Z}/(q_j\mathbb{Z}))\right)\oplus C_i(X,\mathbb{Z})^m.$$

Explicitly, for every $\sigma\in\overrightarrow{X}(i)$ and every $(a_1,\dots,a_{l+m})\in\mathbb{A}$, we identify $(a_1,\dots,a_{l+m})\mathbb{1}_\sigma \in C_i(X,\mathbb{A})$ with $\bigoplus_{j=1}^{l+m} a_j\mathbb{1}_\sigma \in \left(\bigoplus_{j=1}^l C_i(X,\mathbb{Z}/(q_j\mathbb{Z}))\right)\oplus C_i(X,\mathbb{Z})^m$.

For $1\le j\le l$, let $\varphi_j:\mathbb{Z}\to\mathbb{Z}/(q_j\mathbb{Z})$ be the quotient map. For every $-1\le i\le n$, φ_j extends to a surjective homomorphism $\varphi_j : C_i(X,\mathbb{Z})\to C_i(X,\mathbb{Z}/(q_j\mathbb{Z}))$ (this is a homomorphism of $\mathbb{Z}$-modules).

Let $\mathrm{Cone}_X = S$ be a $(k,\mathbb{Z})$-cone function. For every $1 \leq j \leq l$, $-1 \leq i \leq k$ we define the function

$$S_i^{(j)} : C_i(X;\mathbb{Z}/(q_j\mathbb{Z})) \to C_{i+1}(X;\mathbb{Z}/(q_j\mathbb{Z}))$$

as follows:

$$S_i^{(j)}(\varphi_j(A)) = \varphi_j(S_i(A)), \forall -1 \leq i \leq k, \forall A \in C_i(X;\mathbb{Z}).$$

We note that $S_i^{(j)}$ is well-defined: Let $A, A' \in C_i(X;\mathbb{Z})$ with $\varphi_j(A) = \varphi_j(A')$. Then $A - A' \in \mathrm{Ker}(\varphi_j)$, which implies that there is $A'' \in C_i(X;\mathbb{Z})$ such that $A - A' = q_j A''$. Thus

$$\begin{aligned} S_i^{(j)}(\varphi_j(A)) - S_i^{(j)}(\varphi_j(A')) &= \varphi_j(S_i(A)) - \varphi_j(S_i(A')) \\ &= \varphi_j(S_i(A - A')) = \varphi_j(S_i(q_j A'')) = \varphi_j(q_j S_i(A'')) = 0_{\mathbb{Z}/(q_j\mathbb{Z})} \end{aligned}$$

and it follows that $S_i^{(j)}(\varphi_j(A)) = S_i^{(j)}(\varphi_j(A'))$. Also, by the fact that φ_j is surjective on every $C_i(X,\mathbb{Z}/(q_j\mathbb{Z}))$ it follows that $S_i^{(j)}$ is defined on every $A \in C_i(X;\mathbb{Z}/(q_j\mathbb{Z}))$.

The cone equation for $S^{(j)}$ follows from the fact that $\phi_j(\partial A) = \partial \phi_j(A)$ by linearity, and the fact that S satisfies the cone equation. Thus $S^{(j)} = \oplus_{i=-1}^k S_i^{(j)}$ is a $(k,\mathbb{Z}/q_j\mathbb{Z})$-cone function. Note that for every i, every $\sigma \in \overrightarrow{X}(i)$ and every $a \in \mathbb{Z}/(q_j\mathbb{Z})$ it holds that

$$\mathrm{supp}(S_i^{(j)}(a\mathbb{1}_\sigma)) \subseteq \mathrm{supp}(S_i(\mathbb{1}_\sigma)).$$

For every $l+1 \leq j \leq m+l$, we denote by $S^{(j)}$ a copy of S (which is a $(k,\mathbb{Z})$-cone function). For every $-1 \leq i \leq k$ and every

$$\bigoplus_{j=1}^{l+m} A_j \in \left(\bigoplus_{j=1}^{l} C_i(X,\mathbb{Z}/(q_j\mathbb{Z})) \right) \oplus C_i(X,\mathbb{Z})^m,$$

we define

$$S_i^{\mathbb{A}}(\bigoplus_{j=1}^{l+m} A_j) = \bigoplus_{j=1}^{l+m} S^{(j)}(A_j)$$

yielding a family of functions $S_i^{\mathbb{A}} : C_i(X,\mathbb{A}) \to C_{i+1}(X,\mathbb{A})$.

We conclude the proof by observing that $S^{\mathbb{A}} = \bigoplus_{i=-1}^k S_i^{\mathbb{A}}$ is a $(k,\mathbb{A})$-cone function such that for every $(a_1,\ldots,a_{l+m}) \in \mathbb{A}$, every $-1 \leq i \leq k$ and every $\sigma \in \overrightarrow{X}(i)$, it holds that

$$\mathrm{supp}(S^{\mathbb{A}}((a_1,\ldots,a_{l+m})\mathbb{1}_\sigma)) \subseteq \mathrm{supp}(S(\mathbb{1}_\sigma)).$$

Since $\mathrm{Cone}_X = S$ was an arbitrary $(k,\mathbb{Z})$-cone function, it follows that $\mathrm{Rad}_k(X,\mathbb{A}) \leq \mathrm{Rad}_k(X,\mathbb{Z})$.

$\square$

In light of this proposition, in what follows, we will be interested in constructing $\mathbb{Z}$-cone functions with bounded radius.

Observation 5.2.15. We observe that for every j, $C_j(X;\mathbb{Z})$ is a free $\mathbb{Z}$-module and thus the condition in the definition of the cone function that

$$\mathrm{Cone}_X^j : C_j(X;\mathbb{Z}) \to C_{j+1}(X;\mathbb{Z})$$

is a homomorphism is equivalent to the condition that this map is a $\mathbb{Z}$-linear map of $\mathbb{Z}$-modules.

By this linearity, the cone equation is equivalent to the condition:

$$\partial_{j+1} \mathrm{Cone}_X(\mathbb{1}_\sigma) + \mathrm{Cone}_X(\partial_j \mathbb{1}_\sigma) = \mathbb{1}_\sigma, \forall \sigma \in \overrightarrow{X}(j).$$

In other words, in order to verify that a linear function is a cone function, it is enough to check the cone equation on $\mathbb{1}_\sigma$ for every oriented simplex σ.

Moreover, when one defines a cone function, it is enough to define it as a linear function that fulfills the cone equation on $\mathbb{1}_\sigma$ for σ in a subset of oriented simplices that contain at least one orientation for every simplex.

Combining Theorem 5.2.7 with Proposition 5.2.14 yields the following:

Corollary 5.2.16. Let $n \geq 1$, $\mathcal{R} > 0$ be constants. For every finite pure n-dimensional simplicial complex X, if

- the group $\mathrm{Aut}(X)$ acts transitively on $X(n)$

and

- for $0 \leq k \leq n-1$ it holds that $\mathrm{Rad}_k(X, \mathbb{Z}) \leq \mathcal{R}$

then for every finitely generate abelian group $\mathbb{A}$ we have

$$h^k_{\mathrm{cb}}(X, \mathbb{A}) \geq \frac{1}{\mathcal{R}\binom{n+1}{k+1}}.$$

5.3 Cone functions for joins

The goal of this section is to show how to construct a cone function for the join of two simplicial complexes given that the underlying complexes admit a cone function.

We denote by $\sqcup$ the disjoint union of sets. Given two simplicial complexes Y_1 and Y_2 with vertex sets $V(Y_1)$ and $V(Y_2)$, the *join* of Y_1 and Y_2, denoted $Y_1 * Y_2$, is the simplicial complex with the vertex set $V(Y_1) \sqcup V(Y_2)$ and simplices $\sigma \sqcup \tau$ for every $\sigma \in Y_1$ and $\tau \in Y_2$. We note that $Y_1 * Y_2$ includes all the simplices of the form $\tau = \emptyset \sqcup \tau$ and $\sigma = \sigma \sqcup \emptyset$ for every $\sigma \in Y_1$ and $\tau \in Y_2$.

In particular, the join of a complex Y with a single vertex v is called a cone, see Figure 5.2.

We again begin by studying the non-abelian case.

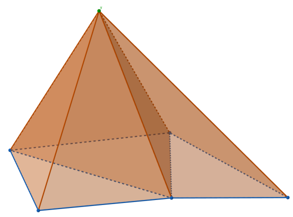


Figure 5.2: This shows an example of a join $Y * \{y\}$, where complex Y is depicted in blue, y in green.

Non-abelian setting We start by considering the join of a non-empty complex with a single vertex.

Proposition 5.3.1. Let Y be a non-empty simplicial complex and $\{w\}$ a complex with only one vertex w. Then there exists a 1-cone function $\mathrm{Cone}_{\{w\}*Y}$ with $\mathrm{Rad}(\mathrm{Cone}_{\{w\}*Y}) \leq 1$.

Proof. Set $X = \{w\} * Y$. First, assume that $\dim Y \geq 1$. We note that for every $(u, v) \in Y_{\mathrm{ord}}(1)$ it holds that $(w, u, v) \in X_{\mathrm{ord}}(2)$ and thus denoting by (w) the trivial loop around w we have

$$(w; u; v; w) \overset{(TR)}{\sim} (w; v; w) \overset{(BT)}{\sim} (w).$$

Hence, we define a function $\mathrm{Cone}_{\{w\}*Y} = (v_0, (P_u)_{u \in X(0)}, (T_{(u,v)})_{(u,v) \in X_{\mathrm{ord}}(1)})$ as follows:

- $v_0 = w$.
- For every $u \in X(0)$, $P_u = (w; u)$.
- For every $(u, v) \in X_{\mathrm{ord}}(1)$, $T_{u,v} = (P_0 = P_u \circ (u; v) \circ P_v^{-1}, P_1 = (w; v; w), P_2 = (w))$.

It follows that $\mathrm{Rad}(\mathrm{Cone}_{\{w\}*Y}) = 1$ as needed.

If $\dim Y = 0$ then $\dim X = 1$. But we can still define a 1-cone function of radius 1: The v_0 and P_u are defined as before. The only edges in $X_{\mathrm{ord}}(1)$ are of the form (w, u) (or (u, w)) with $u \in Y(0)$. Thus $T_{(w,u)}$ consists of $P_0 = (w) \circ (w; u) \circ (u; w) \overset{(BT)}{\sim} P_1 = (w)$ (and $T_{(u,w)}$ of $P_0 = (w; u) \circ (u; w) \circ (w) \overset{(BT)}{\sim} (w)$). $\square$

Remark 5.3.2. We note that in the above construction, the cone radius is still ≤ 1, even if Y has no edges or if Y is disconnected. If Y is empty, we can still say that $\{w\} * \emptyset$ has a 0-cone function, since it will be a connected graph consisting of one single vertex.

Lemma 5.3.3. Let Y_1, Y_2 be non-empty, finite simplicial complexes. Then $Y_1 * Y_2$ admits a 0-cone function with $\mathrm{Rad}_0(\mathrm{Cone}_{Y_1 * Y_2}) \leq 2$.

Proof. We have that the 1-skeleton of $Y_1 * Y_2$ contains the complete bipartite graph on $Y_1(0) \sqcup Y_2(0)$ as a subgraph. In particular, every two vertices are connected by a path of length at most 2. $\square$

Proposition 5.3.4. Let Y_1 be an n-dimension simplicial complex, $n \geq 1$, with 0-cone function $\mathrm{Cone}_{Y_1} = (v_0^1, (P_u^1)_{u \in Y_1(0)})$. Let Y_2 be a non-empty, finite simplicial complex. Then $Y_1 * Y_2$ admits a 1-cone function $\mathrm{Cone}_{Y_1 * Y_2}$ with $\mathrm{Rad}_0(\mathrm{Cone}_{Y_1*Y_2}) = \mathrm{Rad}_0(\mathrm{Cone}_{Y_1})$ and $\mathrm{Rad}_1(\mathrm{Cone}_{Y_1*Y_2}) \leq 2\,\mathrm{Rad}_0(\mathrm{Cone}_{Y_1}) + 1$.

Proof. We want to define a 1-cone function $(v_0, (P_u)_{u \in Y_1(0) \sqcup Y_2(0)}, (T_{(u,v)})_{(u,v) \in (Y_1 * Y_2)_{\mathrm{ord}}(1)})$ for $Y_1 * Y_2$. First, note that $(Y_1 * Y_2)_{\mathrm{ord}}(1) = Y_{1\mathrm{ord}}(1) \sqcup Y_{2\mathrm{ord}}(1) \sqcup \{(u,v) \mid u \in Y_i, v \in Y_j, \{i,j\} = \{1,2\}\}$.

- We set $v_0 = v_0^1$.
- For $u \in Y_1(0)$, we set $P_u = P_u^1$.
- For $u \in Y_2(0)$, we set $P_u = (v_0; u)$. (Note that this is well defined since $v_0 \in Y_1$ is connected by an edge to any vertex of Y_2.)
- For $(u,v) \in Y_{2\mathrm{ord}}(1)$, we set $T_{(u,v)} = (P_0 = (v_0; u; v; v_0), P_1 = (v_0; v; v_0), P_2 = (v_0))$ with $P_0 \overset{(TR)}{\sim} P_1$ and $P_1 \overset{(BT)}{\sim} P_2$.
- Fix $u \in Y_1(0), v \in Y_2(0)$ and let $P_u = (v_0; u_1; \ldots; u_n; u)$. Then we define $T_{(u,v)}$ as follows, see also Figure 5.3.

$$\begin{aligned}
P_0 &= (v_0; u_1; \ldots; u_n) \circ (u_n; u) \circ (u; v) \circ (v; v_0) \\
&\qquad \wr_{(TR)} \text{ with triangle } (u_n, u, v) \\
P_1 &= (v_0; u_1; \ldots; u_n) \circ (u_n; v) \circ (v; v_0) \\
&\qquad \wr_{(TR)} \\
&\qquad \vdots \\
&\qquad \wr_{(TR)} \\
P_n &= (v_0; u_1) \circ (u_1; v) \circ (v; v_0) \\
&\qquad \wr_{(TR)} \text{ with triangle } (v_0, u_1, v) \\
&(v_0; v) \circ (v; v_0) \\
&\qquad \wr_{(BT)} \\
&\qquad (v_0)
\end{aligned}$$

  Observe that $|T_{(u,v)}| = \mathrm{length}(P_u)$.

- Lastly, fix $(u,v) \in Y_{1\mathrm{ord}}(1)$ and denote the paths from v_0 to u and v by $P_u = (v_0; u_1; \ldots; u_n; u), P_v = (v_0; v_1; \ldots; v_n; v)$, respectively. Then we define $T_{(u,v)}$ as follows, similar to the above case, but now doing the contractions on "both sides" of

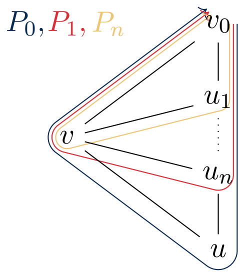


Figure 5.3: Visualization of the paths in $T_{(u,v)}$ for $u \in Y_1(0), v \in Y_2(0)$.

the path. We need to fix an arbitrary vertex $x \in Y_2(0)$. See also Figure 5.4.

$$
\begin{aligned}
P_0 &= (v_0; u_1; \ldots; u_n; u) \circ (u; v) \circ (v; v_n; \ldots; v_1; v_0) \\
&\quad \wr_{(TR)} \text{ with triangle } (u, x, v) \\
P_1 &= (v_0; u_1; \ldots; u_n) \circ (u_n; u; x) \circ (x; v; v_n) \circ (v_n; \ldots; v_1; v_0) \\
&\quad \wr_{(TR)} \text{ with triangle } (u_n, u, x) \text{ and } (x, v, v_n) \\
P_3 &= (v_0; u_1; \ldots; u_n) \circ (u_n; x) \circ (x; v_n) \circ (v_n; \ldots; v_0) \\
&\quad \vdots \\
&(v_0; u_1; x) \circ (x; v_1; v_0) \\
&\quad \wr_{(TR)} \text{ with triangle } (v_0, u_1, x) \text{ and } (x, v_1, v_0) \\
&(v_0; x) \circ (x; v_0) \\
&\quad \wr_{(BT)} \\
&(v_0)
\end{aligned}
$$

Observe that the paths P_u, P_v don't necessary have to be of the same length. In that

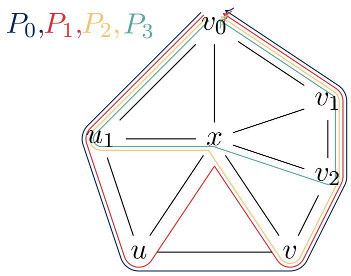


Figure 5.4: Visualization of the paths in $T_{(u,v)}$ for $(u, v) \in Y_{1\text{ord}}(1)$ (for an example with $\text{length}(P_u) = 2, \text{length}(P_v) = 3$).

case, the induction on one side halts earlier then on the other side. Counting the total number of (TR) operations made, we have $|T_{(u,v)}| = \text{length}(P_u) + \text{length}(P_v) + 1$.

$\square$

Proposition 5.3.5. Let Y_1, Y_2 be finite, non-empty, simplicial complexes. Assume that there is a 1-cone function Cone_{Y_1} with finite cone radius. Then there is a 1-cone function $\mathrm{Cone}_{Y_1 * Y_2}$ such that,

$$\begin{aligned}\mathrm{Rad}_0(\mathrm{Cone}_{Y_1*Y_2}) &= \mathrm{Rad}_0(\mathrm{Cone}_{Y_1})\\ \mathrm{Rad}_1(\mathrm{Cone}_{Y_1*Y_2}) &= \max\{\mathrm{Rad}_0(\mathrm{Cone}\,Y_1), \mathrm{Rad}_1(\mathrm{Cone}_{Y_1}), 1\}.\end{aligned}$$

Proof. Recall that $(Y_1 * Y_2)(0) = Y_1(0) \sqcup Y_2(0)$ and that any simplex of either Y_i can be seen as a simplex in the join as well. In particular,

$$(Y_1 * Y_2)_{\mathrm{ord}}(1) = Y_{1\mathrm{ord}}(1) \sqcup Y_{2\mathrm{ord}}(1) \sqcup \{(u,v) \mid u \in Y_i(0), v \in Y_j(0), i,j \in \{1,2\}, i \neq j\}.$$

Let $\mathrm{Cone}_{Y_1} = (v_0^1, (P_u^1)_{u \in Y_1(0)}, (T^1_{(u,v)})_{(u,v)\in Y_{1\mathrm{ord}}(1)})$. We will define a cone function for the join $\mathrm{Cone}_{Y_1*Y_2} = (v_0, (P_u)_{u \in Y_1 * Y_2(0)}, (T_{(u,v)})_{(u,v)\in (Y_1*Y_2)_{\mathrm{ord}(1)}})$ as follows.

- We set $v_0 = v_0^1$.
- For $u \in Y_1(0)$, set $P_u = P_u^1$.
- For $v \in Y_2(0)$, set $P_v = (v_0; v)$.
- For $(u,v) \in Y_{1\mathrm{ord}}(1)$, set $T_{(u,v)} = T^1_{(u,v)}$.
- For $(u,v) \in Y_{2\mathrm{ord}}(1)$, set $T_{(u,v)} = (P_0 = (v_0;u) \circ (u;v) \circ (v;v_0), P_1 = (v_0;v) \circ (v;v_0), P_2 = (v_0))$. Note that $P_0 \overset{(TR)}{\sim} P_1$ and $P_1 \overset{(BT)}{\sim} P_2$.
- Lastly, we define the most involved case, where $u \in Y_1(0), v \in Y_2(0)$. We have $P_0 = P_u \circ (u;v) \circ (v;v_0)$. We give names to the vertices in P_u: $P_u = (v_0; u_1; u_2; \ldots; u_n; u)$. Note that each u_i is also connected to v by an edge. Next we define $P_1 = (v_0; u_1; \ldots; u_n) \circ (u_n; v) \circ (v; v_0)$ and observe that $P_0 \overset{(TR)}{\sim} P_1$ via the triangle (u_n, u, v). We continue this process iteratively, see also Figure 5.5: $P_i = (v_0; u_1; \ldots; u_{n-i+1}) \circ (u_{n-i+1}; v) \circ (v; v_0)$, hence $P_{i-1} \overset{(TR)}{\sim} P_i$ via the triangle $(u_{n-i+1}, u_{n-i+2}, v)$ for $i = 1, \ldots, n$. Then $P_n = (v_0; u_1; v; v_0)$, and we finish the contraction via $P_{n+1} = (v_0; v; v_0), P_{n+2} = (v_0)$ with $P_n \overset{(TR)}{\sim} P_{n+1}$ and $P_{n+1} \overset{(BT)}{\sim} P_{n+2}$.

  In particular, we have used (TR) once for every edge in P_u, hence $|T_{(u,v)}| = \mathrm{length}(P_u)$.

  The definition of $T_{(v,u)}$ is analogous, just with reversed direction of the paths.

□

To avoid having to make case distinctions between cone functions for 1-dimensional complexes and complexes with dimension ≥ 2, we introduce the following notation.

Definition 5.3.6. Let X be a finite simplicial complex. We say that X has a non-abelian cone function (NAC) if

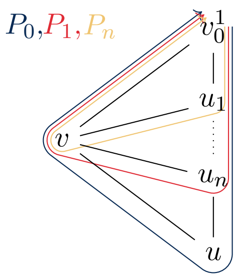


Figure 5.5: Visualization of the paths in $T_{(u,v)}$ for $u \in Y_1(0), v \in Y_2(0)$.

- $X \neq \emptyset$,
- if $\dim X = 1$ there exists a 0-cone function for X,
- if $\dim X \geq 2$ there exists a 1-cone function for X.

Furthermore, we define the cone radius of X, denoted $\mathrm{Rad}(X)$ as

- if $\dim X = 0, X \neq \emptyset$ then $\mathrm{Rad}(X) = 1$,
- if $\dim X = 1$ and X has a NAC then
$$\mathrm{Rad}(X) = \min\{\mathrm{Rad}(\mathrm{Cone}_X) \mid \mathrm{Cone}_X \text{ a 0-cone function of } X\},$$
- if $\dim X \geq 2$ and X has a NAC then
$$\mathrm{Rad}(X) = \min\{\mathrm{Rad}(\mathrm{Cone}_X) \mid \mathrm{Cone}_X \text{ a 1-cone function of } X\},$$
- if X does not admit a NAC, we set $\mathrm{Rad}(X) = \infty$.

With this notation, we can summarize the results of this chapter. By summarizing the different cases, we loose some sharpness of the bounds on the cone radius.

Corollary 5.3.7. Let X, Y be two finite simplicial complexes. If both X and Y have a NAC then $X * Y$ has a NAC and $\mathrm{Rad}(X * Y) \leq 2\min\{\mathrm{Rad}(X), \mathrm{Rad}(Y)\} + 1$.

Proof. If $\dim X = \dim Y = 0$ then $\dim X * Y = 1$ and it has a 0-cone function by Lemma 5.3.3. If $\dim X = 1$ then the assumption implies that X has a 0-cone function. Thus Proposition 5.3.4 gives the desired result. If $\dim X \geq 2$, then by assumption there exists a 1-cone function for X. Hence Proposition 5.3.5 gives the desired result. Using the same argument for Y, we can in total use the minium of the two radii. □

Abelian setting Our goal is to obtain similar results for cone functions of joins as in the non-abelian setting. We will here only consider cones with coefficients in $\mathbb{Z}$, as suggested by Proposition 5.2.14.

We start by establishing the necessary technical tools.

Definition 5.3.8. Let X be a simplicial complex, $\tilde{\sigma} = \{v_0, \ldots, v_k\} \in X$ and $\tilde{\tau} = \{u_0, \ldots, u_\ell\} \in \mathrm{lk}_X(\sigma)$, i.e. $\tilde{\tau} \cap \tilde{\sigma} = \emptyset, \tilde{\tau} \cup \tilde{\sigma} \in X$. Let $\sigma = [v_0, \ldots, v_k], \tau = [u_0, \ldots, u_k]$. Then we define

$$[\sigma, \tau] = [v_0, \ldots, v_k, u_0, \ldots, u_\ell] \in X_{\mathrm{ord}}(k + l + 1)$$

and

$$[\mathbb{1}_\sigma, \mathbb{1}_\tau] = \mathbb{1}_{[\sigma,\tau]}.$$

Lemma 5.3.9. Let $X = Y_1 * Y_2$, $\sigma \in Y_{1\mathrm{ord}}(k)$ and $\tau \in Y_{2\mathrm{ord}}(\ell)$, then $[\sigma, \tau]$ is always well defined in $Y_1 * Y_2$. Furthermore, we have

$$\partial[\mathbb{1}_\sigma, \mathbb{1}_\tau] = [\partial\mathbb{1}_\sigma, \mathbb{1}_\tau] + (-1)^{k+1}[\mathbb{1}_\sigma, \partial\mathbb{1}_\tau].$$

We extend the $[\cdot, \cdot]$ operation bilinearly to chains $A \in C_k(Y_1, \mathbb{Z}), B \in C_\ell(Y_2, \mathbb{Z})$, i.e. for $A = \sum_i \lambda_i \mathbb{1}_{\sigma_i}, B = \sum_j \mu_j \mathbb{1}_{\tau_j}$ we have

$$[A, B] = \sum_{i,j} \lambda_i \mu_j [\mathbb{1}_{\sigma_i}, \mathbb{1}_{\tau_j}].$$

The linearity of ∂ then implies

$$\partial[A, B] = [\partial A, B] + (-1)^{k+1}[A, \partial B].$$

Proof. First of all, note that if $\sigma \in Y_1, \tau \in Y_2$, then we always have $\tau \in \mathrm{lk}_{Y_1*Y_2}(\sigma)$ and $\sigma \in \mathrm{lk}_{Y_1*Y_2}(\tau)$.

Let $\sigma = [v_0, \ldots, v_k], \tau = [u_0, \ldots, u_\ell]$. Then

$$\begin{aligned}
\partial[\mathbb{1}_\sigma, \mathbb{1}_\tau] &= \partial\mathbb{1}_{[v_0,\ldots,v_k,u_0,\ldots,u_\ell]} \\
&= \sum_{i=0}^{k} (-1)^i \mathbb{1}_{[v_0,\ldots,v_{i-1},v_{i+1},\ldots,v_k,u_0,\ldots,u_\ell]} + \sum_{i=0}^{\ell} (-1)^{k+1+i} \mathbb{1}_{[v_0,\ldots,v_k,u_0,\ldots,u_{i-1},u_{i+1},\ldots,u_\ell]} \\
&= [\partial\mathbb{1}_\sigma, \mathbb{1}_\tau] + (-1)^{k+1}[\mathbb{1}_\sigma, \partial\mathbb{1}_\tau].
\end{aligned}$$

□

Also note that $\mathrm{supp}([A, B]) = \mathrm{supp}(A) \times \mathrm{supp}(B)$.

We can now bound the cone radius of the join of a complex with a single vertex.

Proposition 5.3.10. Let Y be a non-empty finite simplicial complex and $\{v\}$ be a complex with only one vertex v. Then for every $k \in \mathbb{N} \cup \{0\}$, there is a k-cone function $\mathrm{Cone}_{\{v\}*Y}$ such that $\mathrm{Rad}(\mathrm{Cone}_{\{v\}*Y}) \leq 1$.

Proof. Fix $k \in \mathbb{N} \cup \{0\}$ and define a function $\mathrm{Cone}_{\{v\}*Y}$ as follows:

- $\mathrm{Cone}_{\{v\}*Y}(\emptyset) = \mathbb{1}_{[v]}$.
- For every $0 \leq j \leq k$ and every $[v_0, \ldots, v_j] \in \overrightarrow{Y}(j)$,

$$\mathrm{Cone}_{\{v\}*Y}(\mathbb{1}_{[v_0,\ldots,v_j]}) = [\mathbb{1}_{[v]}, \mathbb{1}_{[v_0,\ldots,v_j]}] = \mathbb{1}_{[v,v_0,\ldots,v_j]}.$$

- For every $0 \leq j \leq k$ and every $\{v_1, \ldots, v_j\} \in Y$,

$$\mathrm{Cone}_{\{v\}*Y}(\mathbb{1}_{[v,v_1,\ldots,v_j]}) = 0.$$

We then extend the function $\mathrm{Cone}_{\{v\}*Y}$ $\mathbb{Z}$-linearly.

We check that the cone equation holds for the above function. In the case of $[v_0, \ldots, v_j] \in \overrightarrow{Y}(j)$ this is due to Lemma 5.3.9. For $\{v_1, \ldots, v_j\} \in Y$, we note that

$$\begin{aligned}
&\partial_{j+1} \mathrm{Cone}_{\{v\}*Y}(\mathbb{1}_{[v,v_1,\ldots,v_j]}) + \mathrm{Cone}_{\{v\}*Y}(\partial_j \mathbb{1}_{[v,v_1,\ldots,v_j]}) \\
=&0 + \mathrm{Cone}_{\{v\}*Y}(\mathbb{1}_{[v_1,\ldots,v_j]}) - \mathrm{Cone}_{\{v\}*Y}([\mathbb{1}_{[v]}, \partial_{j-1} \mathbb{1}_{[v_1,\ldots,v_j]}]) \\
=&0 + \mathbb{1}_{[v,v_1,\ldots,v_j]} + 0 = \mathbb{1}_{[v,v_1,\ldots,v_j]}
\end{aligned}$$

as needed. Thus, by Observation 5.2.15, the above function is a k-cone function and per definition $\mathrm{Rad}(\mathrm{Cone}_{\{v\}*Y}) \leq 1$. □

Remark 5.3.11. We note that in the above construction, the cone function is well-defined for every k, i.e., it is well-defined even in the cases where $k \geq \dim(\{v\} * Y)$.

The following proposition is a reformulation of [Wil22, Lemma 6.15]. We provide a proof that is adapted to our setting.

Proposition 5.3.12. Let Y_1, Y_2 be finite simplicial complexes of dimension n_1, n_2 correspondingly. Assume that for $i = 1, 2$ there is an n_i-cone function Cone_{Y_i} and constants $R_j^{(i)} \in \mathbb{N}, -1 \leq j \leq n_i - 1$ such that $\mathrm{Rad}_j(\mathrm{Cone}_{Y_i}) \leq R_j^{(i)}$. Then there is an $(n_1 + n_2)$-cone function $\mathrm{Cone}_{Y_1 * Y_2}$ such that

- for every $-1 \leq j < n_1$

$$\mathrm{Rad}_j(\mathrm{Cone}_{Y_1*Y_2}) \leq \max_{-1 \leq j_1 \leq j} R_{j_1}^{(1)},$$

- for every $n_1 \leq j \leq n_1 + n_2$

$$\mathrm{Rad}_j(\mathrm{Cone}_{Y_1*Y_2}) \leq \max\{\max_{-1 \leq j_1 \leq n_1 - 1} R_{j_1}^{(1)}, ((n_1 + 1)R_{n_1-1}^{(1)} + 1)R_{j-n_1-1}^{(2)}\}.$$

Proof. Let v be the apex of Cone_{Y_1}. Define

$$\mathrm{Cone}_{Y_1*Y_2}(\mathbb{1}_\emptyset) = \mathbb{1}_{[v]}.$$

Let $-1 \le j_1 \le n_1$ and $-1 \le j_2 \le n_2$ such that $0 \le j_1 + j_2 + 1 \le n_1 + n_2$. For $\sigma_i \in \overrightarrow{X}(j_i), i = 1, 2$, we define

$$\mathrm{Cone}_{Y_1 * Y_2}(\mathbb{1}_{[\sigma_1,\sigma_2]}) = \begin{cases} [\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2}] & \dim(\sigma_1) < n_1 \\ (-1)^{n_1+1}[\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \mathrm{Cone}_{Y_2}(\mathbb{1}_{\sigma_2})] & \dim(\sigma_1) = n_1 \end{cases}$$

and extend it linearly.

We will check that the cone equation is fulfilled. We note that by Lemma 5.3.9 and linearity of the cone function,

$$\mathrm{Cone}_{Y_1*Y_2}(\partial_{j_1+j_2+1}\mathbb{1}_{[\sigma_1,\sigma_2]}) = \mathrm{Cone}_{Y_1*Y_2}([\partial_{j_1}\mathbb{1}_{\sigma_1}, \mathbb{1}_{\sigma_2}]) + (-1)^{j_1+1}\,\mathrm{Cone}_{Y_1*Y_2}([\mathbb{1}_{\sigma_1}, \partial_{j_2}\mathbb{1}_{\sigma_2}]).$$

Thus in case $\dim(\sigma_1) < n_1$ we have

$$\mathrm{Cone}_{Y_1*Y_2}(\partial_{j_1+j_2+1}\mathbb{1}_{[\sigma_1,\sigma_2]}) = [\mathrm{Cone}_{Y_1}(\partial_{j_1}\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2}] + (-1)^{j_1+1}[\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}), \partial_{j_2}\mathbb{1}_{\sigma_2}]$$

and in case $\dim(\sigma_1) = n_1$ we have $\dim \partial\sigma_1 < n_1$ thus applying the definition of the cone function we get

$$\begin{aligned}
&\mathrm{Cone}_{Y_1*Y_2}(\partial_{j_1+j_2+1}\mathbb{1}_{[\sigma_1,\sigma_2]}) \\
&= [\mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2}] + (-1)^{n_1+1}(-1)^{n_1+1}[\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \mathrm{Cone}_{Y_2}(\partial_{j_2}\mathbb{1}_{\sigma_2})] \\
&= [\mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2}] + [\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \mathrm{Cone}_{Y_2}(\partial_{j_2}\mathbb{1}_{\sigma_2})] \overset{\text{cone equ. } Y_1}{=} \\
&= [\mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2}] + [\partial_{n_1+1}\,\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}), \mathrm{Cone}_{Y_2}(\partial_{j_2}\mathbb{1}_{\sigma_2})]
\end{aligned}$$

Next, we look at the other term in the cone equation. Assume first that $\dim(\sigma_1) < n_1$. Then

$$\begin{aligned}
&\partial_{j_1+j_2+2}\,\mathrm{Cone}_{Y_1*Y_2}(\mathbb{1}_{[\sigma_1,\sigma_2]}) \\
&= \partial[\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2}] \overset{\text{Lemma 5.3.9}}{=} \\
&= [\partial_{j_1+1}\,\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2}] + (-1)^{j_1+2}[\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}), \partial_{j_2}\mathbb{1}_{\sigma_2}] \overset{\text{cone equ. } Y_1}{=} \\
&= [\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{j_1}\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2}] + (-1)^{j_1+2}[\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}), \partial_{j_2}\mathbb{1}_{\sigma_2}] \\
&= [\mathbb{1}_{\sigma_1}, \mathbb{1}_{\sigma_2}] - \big([\mathrm{Cone}_{Y_1}(\partial_{j_1}\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2}] + (-1)^{j_1+1}[\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}), \partial_{j_2}\mathbb{1}_{\sigma_2}]\big) \\
&= \mathbb{1}_{[\sigma_1,\sigma_2]} - \mathrm{Cone}_{Y_1*Y_2}(\partial_{j_1+j_2+1}\mathbb{1}_{[\sigma_1,\sigma_2]}).
\end{aligned}$$

Next, assume that $\dim(\sigma_1) = n_1$. Note that in Y_1 we have

$$\partial_{n_1}(\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1})) = \partial_{n_1}\mathbb{1}_{\sigma_1} - (\partial_{n_1}\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1-1}\partial_{n_1}\mathbb{1}_{\sigma_1})) = 0. \tag{5.1}$$

Thus

$$
\begin{aligned}
&\partial_{n_1+j_2+2}\operatorname{Cone}_{Y_1 * Y_2}(\mathbb{1}_{[\sigma_1,\sigma_2]}) = \\
&= \partial_{n_1+j_2+2}(-1)^{n_1+1}[\mathbb{1}_{\sigma_1} - \operatorname{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \operatorname{Cone}_{Y_2}(\mathbb{1}_{\sigma_2})] \overset{\text{Lemma 5.3.9}}{=} \\
&= (-1)^{n_1+1}[\partial_{n_1}(\mathbb{1}_{\sigma_1} - \operatorname{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1})), \operatorname{Cone}_{Y_2}(\mathbb{1}_{\sigma_2})] \\
&\qquad + (-1)^{n_1+1}(-1)^{n_1+1}[\mathbb{1}_{\sigma_1} - \operatorname{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \partial_{j_2+1}\operatorname{Cone}_{Y_2}(\mathbb{1}_{\sigma_2})] \\
&\overset{(5.1)}{=} [\mathbb{1}_{\sigma_1} - \operatorname{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \partial_{j_2+1}\operatorname{Cone}_{Y_2}(\mathbb{1}_{\sigma_2})] \overset{\text{cone equ. } Y_2}{=} \\
&= [\mathbb{1}_{\sigma_1} - \operatorname{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2} - \operatorname{Cone}_{Y_2}(\partial_{j_2}\mathbb{1}_{\sigma_2})] \overset{\text{cone equ. } Y_1}{=} \\
&= [\mathbb{1}_{\sigma_1} - \operatorname{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2}] - [\partial_{n_1+1}\operatorname{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}), \operatorname{Cone}_{Y_2}(\partial_{j_2}\mathbb{1}_{\sigma_2})] \\
&= [\mathbb{1}_{\sigma_1}, \mathbb{1}_{\sigma_2}] - ([\operatorname{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2}] + [\partial_{n_1+1}\operatorname{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}), \operatorname{Cone}_{Y_2}(\partial_{j_2}\mathbb{1}_{\sigma_2})]) \\
&= \mathbb{1}_{[\sigma_1,\sigma_2]} - \operatorname{Cone}_{Y_1 * Y_2}(\partial_{j_1+j_2+1}\mathbb{1}_{[\sigma_1,\sigma_2]}).
\end{aligned}
$$

Hence, the cone equation holds.

Let $0 \leq j \leq n_1 + n_2$. Then every $\sigma \in \overrightarrow{X}(j)$ is of the form $\sigma = [\sigma_1, \sigma_2]$ where $\sigma_i \in \overrightarrow{X}(j_i), i = 1, 2$ and $j_1 + j_2 + 1 = j$. Note that $j_2 \geq -1$ and thus $j_1 \leq j$. In particular, if $j < n_1$, then $j_1 < n_1$.

For such j_1, j_2, it holds that if $j_1 < n_1$, then

$$|\operatorname{supp}(\operatorname{Cone}_{Y_1 * Y_2}(\mathbb{1}_{[\sigma_1,\sigma_2]}))| \leq R^{(1)}_{j_1}.$$

If $j_1 = n_1$, then $j_2 = j - n_1 - 1$ and

$$
\begin{aligned}
|\operatorname{supp}(\operatorname{Cone}_{Y_1 * Y_2}(\mathbb{1}_{[\sigma_1,\sigma_2]}))| &\leq |\operatorname{supp}(\mathbb{1}_{\sigma_1} - \operatorname{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}))| R^{(2)}_{j_2} \\
&\leq ((n_1 + 1)R^{(1)}_{n_1-1} + 1)R^{(2)}_{j-n_1-1}.
\end{aligned}
$$

Combining these bounds yields the bound stated above for $\operatorname{Rad}_j(\operatorname{Cone}_{Y_1 * Y_2})$. □

Remark 5.3.13. Later, we use a simplified statement which only considers the worst possible case and does not consider the dimension. Thus let $R^{(i)} = \max_{-1 \leq j \leq n_i - 1} R^{(i)}_j$, then

$$\operatorname{Rad}(\operatorname{Cone}_{Y_1 * Y_2}) \leq ((n_1 + 1)R^{(1)} + 1)R^{(2)}.$$

This is significantly weaker than the original statement but will be sufficient in many cases.

5.4 Constructing cone functions by adding vertices

Non-abelian setting Next, we show that we can construct a cone function for a complex X, by starting with a given subcomplex for which we have a cone function and adding a set of vertices which is "well-behaved". We start with a technical definition and then provide an example to show the idea. The precise statement is given in Theorem 5.4.3.

Definition 5.4.1. For a complex X, a subcomplex $X' \subseteq X$ is called full, if for every $v_0, \dots, v_k \in X'$, if $\{v_0, \dots, v_k\} \in X$, then $\{v_0, \dots, v_k\} \in X'$.

Example 5.4.2. Let X be a graph, and X' a full subgraph with known diameter. Furthermore, let W be a set of vertices of X such that $W \cap X'(0) = \emptyset$ and any two vertices $w_1, w_2 \in W$ are not connected by an edge in X, i.e. $\{w_1, w_2\} \notin X$. We further assume that $\mathrm{lk}_X(w) \cap X' \neq \emptyset$ for all $w \in W$. This means that for each $w \in W$ there is an edge in X connecting w to some vertex in X'. Then the full subgraph of X spanned by $X'(0) \cup W$ is connected and its diameter is at most $\mathrm{diam} X' + 2$ (and hence the radius increases by 1). This is visualized in Figure 5.6.

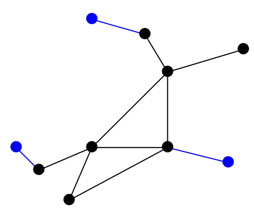


Figure 5.6: The blue vertices are the elements of W and the black graph is X'.

Theorem 5.4.3. Let X be an n-dimensional simplicial complex, $n \geq 2$, and let X' be a full subcomplex. Assume there is a 1-cone function $\mathrm{Cone}_{X'}$ and constants $R'_j \in \mathbb{N}, j = 0, 1$ such that for every $j = 0, 1$, $\mathrm{Rad}_j(\mathrm{Cone}_{X'}) \leq R'_j$. Also let $W \subseteq X(0)$ be a set of vertices such that the following holds:

(i) $W \cap X' = \emptyset$.

(ii) For every $w_1, w_2 \in W$, $\{w_1, w_2\} \notin X(1)$.

(iii) For every $w \in W$, $\mathrm{lk}_X(w) \cap X'$ is a non-empty simplicial complex.

(iv) There are 0-cone functions $\mathrm{Cone}_{\mathrm{lk}_X(w) \cap X'}$ for all $w \in W$ and a constant $R'' \in \mathbb{N}$ such that for every $w \in W$, $\mathrm{Rad}_0(\mathrm{Cone}_{\mathrm{lk}_X(w) \cap X'}) \leq R''$.

Let $X' \cup W$ be the full subcomplex of X spanned by $X'(0) \cup W$. Then there is a 1-cone function $\mathrm{Cone}_{X' \cup W}$ such that $\mathrm{Rad}_0(\mathrm{Cone}_{X' \cup W}) \leq R'_0 + 1, \mathrm{Rad}_1(\mathrm{Cone}_{X' \cup W}) \leq R''(R'_1 + 1)$.

Proof. First, we fix some notation: for each $w \in W$ let $L_w := \mathrm{lk}_X(w) \cap X'$ and

$$\begin{aligned}\mathrm{Cone}_{X'} &= \left(v'_0, (P'_u)_{u \in X'(0)}, \left(T'_{(u,v)}\right)_{(u,v) \in X'_{\mathrm{ord}}(1)}\right),\\ \mathrm{Cone}_{L_w} &= (v^w_0, (P^w_u)_{u \in L_w(0)}).\end{aligned}$$

Observe that $L_w(0)$ are those vertices of X' which are neighbors of w, that $(X' \cup W)(0) = X'(0) \cup W$ and that edges in $X' \cup W$ are all edges of X' or $\{\{u, w\} \mid w \in W, u \in L_w\}$. In particular, there is no edge between two vertices of W.

We define $\mathrm{Cone}_{X' \cup W} = (v_0, (P_u)_{u \in X'(0) \cup W}, (T_{(u,v)})_{(u,v) \in (X' \cup W)_{\mathrm{ord}}(1)})$ as follows.

- We set $v_0 = v_0' \in X'$.
- For $u \in X'(0)$, we set $P_u = P_u'$.
- For $w \in W$, we set $P_w = P'_{v_0^w} \circ (v_0^w; w)$.
- For $(u, v) \in {X_{\mathrm{ord}}}'(1)$, we set $T_{(u,v)} = T'_{(u,v)}$.
- Lastly, we define the most involved case. Let $w \in W, u \in L_w$ (hence $(u, w) \in (X' \cup W)_{\mathrm{ord}}(1)$). Then we define $T_{(u,w)}$ as follows (and $T_{(w,u)}$ similarly by reversing all paths). We have $P_0 = P_u' \circ (u; w) \circ (w; v_0^w) \circ (P'_{v_0^w})^{-1}$. We give names to the vertices in $P_u^w = (v_0^w; u_1; \ldots; u_n; u)$. We get the following series of contractions, see also Figure 5.7.

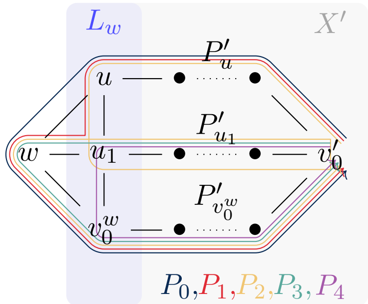


Figure 5.7: Visualization of $T_{(u,w)}$ shown in the case length$(P_u^w) = 2$.

$$\begin{aligned}
P_0 = P_u \circ (u;w) \circ P_w^{-1} & \\
&\wr_{(TR)} \text{ with triangle } (u, u_n, w) \\
P_u \circ (u;u_n) \circ (u_n;w) \circ P_w^{-1} & \\
&\wr_{(BT)} \text{ as often as necessary} \\
P_u \circ (u;u_n) \circ P_{u_n}^{-1} \circ P_{u_n} \circ (u_n;w) \circ P_w^{-1} & \\
&\wr \text{ apply sequence of relations from } T'_{(u,u_n)} \\
(v_0) \circ P_{u_n} \circ (u_n;w) \circ P_w^{-1} & \\
&\wr_{(TR)} \text{ with triangle } (u_n, u_{n-1}, w) \\
P_{u_n} \circ (u_n;u_{n-1}) \circ (u_{n-1};w) \circ P_w^{-1} & \\
&\wr_{(BT)} \text{ as often as necessary} \\
P_{u_n} \circ (u_n;u_{n-1}) \circ P_{u_{n-1}}^{-1} \circ P_{u_{n-1}} \circ (u_{n-1};w) \circ P_w^{-1} & \\
&\wr \text{ apply sequence of relations from } T'_{(u_n,u_{n-1})} \\
(v_0) \circ P_{u_{n-1}} \circ (u_{n-1};w) \circ P_w^{-1} & \\
&\vdots \\
P_{u_1} \circ (u_1;w) \circ P_w^{-1} & \\
&\wr_{(TR)} \text{ with triangle } (u_1, v_0^w, w) \\
P_{u_1} \circ (u_1;v_0^w) \circ (v_0^w;w) \circ P_w^{-1} & \\
&\wr_{(BT)} \text{ as often as necessary} \\
P_{u_1} \circ (u_1;v_0^w) \circ P_{v_0^w}^{-1} P_{v_0^w} (v_0^w;w) P_w^{-1} & \\
&\wr \text{ apply sequence of relations from } T'_{(u_1,v_0^w)} \\
(v_0) \circ P_{v_0^w} \circ (v_0^w;w) \circ (w;v_0^w) P_{v_0^w}^{-1} & \\
&\wr_{(BT)} \text{ as often as necessary} \\
&(v_0)
\end{aligned}$$

In total, for each edge in P_u^w we have used at most $R_1' + 1$ triangle relations. Hence in total $|T_{(u,w)}| \leq R''(R_1' + 1)$.

□

We need a slightly adapted version for 1-dimensional complexes.

Lemma 5.4.4. Let X be a finite, one-dimensional simplicial complex, and let X' be a full subcomplex. Assume there is a 0-cone function $\mathrm{Cone}_{X'}$ and constant $R' \in \mathbb{N}$ such that $\mathrm{Rad}_0(\mathrm{Cone}_{X'}) \leq R'$. Also let $W \subseteq X(0)$ be a set of vertices such that the following holds:

(i) $W \cap X' = \emptyset$.

(ii) For every $w_1, w_2 \in W$, $\{w_1, w_2\} \notin X(1)$.

(iii) For every $w \in W$, $\mathrm{lk}_X(w) \cap X'$ is non-empty.

Let $X' \cup W$ be the full subcomplex of X spanned by $X'(0) \cup W$. Then there is a 0-cone function $\mathrm{Cone}_{X' \cup W}$ such that $\mathrm{Rad}_0(\mathrm{Cone}_{X' \cup W}) \leq R' + 1$

Proof. Let $\mathrm{Cone}_{X'} = (v_0', (P_u')_{u \in X'(0)})$. We define $\mathrm{Cone}_{X' \cup W} = (v_0, (P_u)_{u \in X'(0) \cup W})$ as follows.

- Set $v_0 = v_0'$.
- For $u \in X'(0)$, set $P_u = P_u'$.
- For $w \in W$, pick a vertex $v \in \mathrm{lk}_X(w) \cap X'$, and set $P_w = P_v \circ (v; w)$.

This indeed defines the desired 0-cone function. □

We can summarize both results in the following corollary. Note here that a 0-dimensional complex has a 0-cone function if and only if it contains exactly one vertex.

Corollary 5.4.5. Let X be a finite simplicial complex of dimension at least 1 and let X' be a full subcomplex of dimension at least 1. Assume that X' has a NAC and $\mathrm{Rad}(X') \leq R'$ for some constant $R' \in \mathbb{N}$. Let $W \subseteq X(0)$ be a set of vertices and $R'' \in \mathbb{N}$ such that

(i) $W \cap X' = \emptyset$.

(ii) For every $w_1, w_2 \in W$, $\{w_1, w_2\} \notin X(1)$.

(iii) For every $w \in W$, $\mathrm{lk}_X(w) \cap X'$ has a 0-cone function $\mathrm{Cone}_{\mathrm{lk}_X(w) \cap X'}$ with

$$\mathrm{Rad}_0(\mathrm{Cone}_{\mathrm{lk}_X(w) \cap X'}) \leq R''.$$

Let $X' \cup W$ be the full subcomplex of X spanned by $X'(0) \cup W$. Then $X' \cup W$ has a NAC and $\mathrm{Rad}(X' \cup W) \leq R''(R' + 1)$.

Abelian setting We now also want to establish a way of constructing a cone function by adding vertices to a subcomplex with already known cone function in the abelian setting. The result that we get can be thought of as a special version of the Mayer-Vietoris theorem for cone functions.

Theorem 5.4.6. Let X be an n-dimensional simplicial complex and X' be a full subcomplex. Assume there is an $(n-1)$-cone function $\mathrm{Cone}_{X'}$ and constants $R_j' \in \mathbb{N}, -1 \leq j \leq n-1$ such that for every $-1 \leq j \leq n-1$, $\mathrm{Rad}_j(\mathrm{Cone}_{X'}) \leq R_j'$. Also let $W \subseteq X(0)$ be a set of vertices such that the following holds:

(i) $W \cap X' = \emptyset$.

(ii) For every $w_1, w_2 \in W$, $\{w_1, w_2\} \notin X(1)$.

(iii) For every $w \in W$, $\mathrm{lk}_X(w) \cap X'$ is a non-empty simplicial complex.

(iv) There are $(n-2)$-cone functions $\mathrm{Cone}_{\mathrm{lk}_X(w) \cap X'}$ for all $w \in W$ and constants $R''_j \in \mathbb{N}, -1 \leq j \leq n-2$ such that for every $w \in W$ and every $-1 \leq j \leq n-2$, $\mathrm{Rad}_j(\mathrm{Cone}_{\mathrm{lk}_X(w) \cap X'}) \leq R''_j$.

Let $X' \cup W$ be the full subcomplex of X spanned by $X'(0) \cup W$. Then there is an $(n-1)$-cone function $\mathrm{Cone}_{X' \cup W}$ such that for every $-1 \leq j \leq n-1$, $\mathrm{Rad}_j(\mathrm{Cone}_{X' \cup W}) \leq R''_{j-1}(R'_j + 1)$.

Proof. We will define $\mathrm{Cone}_{X' \cap W}$ by "adding" the cone functions of $\mathrm{lk}_X(w) \cap X', w \in W$ to $\mathrm{Cone}_{X'}$. We will use the notation $L_w = \mathrm{lk}_X(w) \cap X'$.

We note that by linearity and Observation 5.2.15, it is enough to define the cone function on every $\mathbb{1}_{[v_0,\ldots,v_j]}$ for every $[v_0, \ldots, v_j] \in \overrightarrow{X' \cup W}(j)$, $0 \leq j \leq n-1$. We also note that by our assumptions, every $[v_0, \ldots, v_j] \in \overrightarrow{X' \cup W}(j)$ is either in $\overrightarrow{X'}(j)$ or that it has a single vertex in W. Thus, it is enough to define the cone function on $\mathbb{1}_{[v_0,\ldots,v_j]}$ in the following two cases: either $[v_0, \ldots, v_j] \in \overrightarrow{X'}(j)$ or $v_0 = w \in W$ and $[v_1, \ldots, v_j] \in \overrightarrow{L_w}(j-1)$.

We define an $(n-1)$-cone function as a linear continuation of the following: First, we define $\mathrm{Cone}_{X' \cup W}(\emptyset) = \mathbb{1}_{[v]}$ where v is the apex of the cone function $\mathrm{Cone}_{X'}$. Second, for every $0 \leq j \leq n-1$ and every $[v_0, \ldots, v_j] \in \overrightarrow{X'}(j)$, we define

$$\mathrm{Cone}_{X' \cup W}(\mathbb{1}_{[v_0,\ldots,v_j]}) = \mathrm{Cone}_{X'}(\mathbb{1}_{[v_0,\ldots,v_j]}).$$

Last, for every $0 \leq j \leq n-1$, every $w \in W$ and every $[v_1, \ldots, v_j] \in \overrightarrow{L_w}(j-1)$, we define

$$\mathrm{Cone}_{X' \cup W}(\mathbb{1}_{[w,v_1,\ldots,v_j]}) = \mathrm{Cone}_{X'}(\mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\ldots,v_j]})) - [\mathbb{1}_{[w]}, \mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\ldots,v_j]})].$$

We need to verify that the cone equation holds for every $\mathbb{1}_{[w,v_1,\ldots,v_j]}$. We will start by computing $\mathrm{Cone}_{X' \cup W}(\partial_j \mathbb{1}_{[w,v_1,\ldots,v_j]})$:

$$\begin{aligned}
&\mathrm{Cone}_{X' \cup W}(\partial_j \mathbb{1}_{[w,v_1,\ldots,v_j]}) \overset{\text{Lemma 5.3.9}}{=} \\
&= \mathrm{Cone}_{X' \cup W}(\mathbb{1}_{[v_1,\ldots,v_j]}) - \mathrm{Cone}_{X' \cup W}([\mathbb{1}_{[w]}, \partial_{j-1} \mathbb{1}_{[v_1,\ldots,v_j]}]) \\
&= \mathrm{Cone}_{X'}(\mathbb{1}_{[v_1,\ldots,v_j]}) - \mathrm{Cone}_{X'}(\mathrm{Cone}_{L_w}(\partial_{j-1} \mathbb{1}_{[v_1,\ldots,v_j]})) \\
&\qquad + [\mathbb{1}_{[w]}, \mathrm{Cone}_{L_w}(\partial_{j-1} \mathbb{1}_{[v_1,\ldots,v_j]})] \overset{\text{cone equ.} L_w}{=} \\
&= \mathrm{Cone}_{X'}(\mathbb{1}_{[v_1,\ldots,v_j]}) - \mathrm{Cone}_{X'}(\mathbb{1}_{[v_1,\ldots,v_j]}) + \mathrm{Cone}_{X'}(\partial_j \mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\ldots,v_j]})) \\
&\qquad + [\mathbb{1}_{[w]}, \mathrm{Cone}_{L_w}(\partial_{j-1} \mathbb{1}_{[v_1,\ldots,v_j]})] \\
&= \mathrm{Cone}_{X'}(\partial_j \mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\ldots,v_j]})) + [\mathbb{1}_{[w]}, \mathrm{Cone}_{L_w}(\partial_{j-1} \mathbb{1}_{[v_1,\ldots,v_j]})].
\end{aligned}$$

Using this computation, we will verify the cone equation for $\mathrm{Cone}_{X'\cup W}$ (below we use the cone equation for $\mathrm{Cone}_{X'}$ and Cone_{L_w} without noting it):

$$\begin{aligned}
&\partial_{j+1}\,\mathrm{Cone}_{X'\cup W}(\mathbb{1}_{[w,v_1,\dots,v_j]}) = \\
&= \partial_{j+1}\,\mathrm{Cone}_{X'}(\mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\dots,v_j]})) - \partial_{j+1}[\mathbb{1}_{[w]},\mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\dots,v_j]})] \overset{\text{cone equ.}X'}{=} \\
&= \mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\dots,v_j]}) - \mathrm{Cone}_{X'}(\partial_j\,\mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\dots,v_j]})) \\
&\qquad - \partial_{j+1}[\mathbb{1}_{[w]},\mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\dots,v_j]})] \overset{\text{Lemma 5.3.9}}{=} \\
&= -\,\mathrm{Cone}_{X'}(\partial_j\,\mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\dots,v_j]})) + \mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\dots,v_j]}) \\
&\qquad - \Big(\mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\dots,v_j]}) - [\mathbb{1}_{[w]},\partial_j\,\mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\dots,v_j]})]\Big) \overset{\text{cone equ.}L_w}{=} \\
&= -\,\mathrm{Cone}_{X'}(\partial_j\,\mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\dots,v_j]})) + [\mathbb{1}_{[w]},\mathbb{1}_{[v_1,\dots,v_j]} - \mathrm{Cone}_{L_w}(\partial_{j-1}\mathbb{1}_{[v_1,\dots,v_j]})] \\
&= -\,\mathrm{Cone}_{X'\cup W}(\partial_j\mathbb{1}_{[w,v_1,\dots,v_j]}) + \mathbb{1}_{[w,v_1,\dots,v_j]}
\end{aligned}$$

as needed.

In order to conclude the proof, we will show that $\mathrm{Rad}_j(\mathrm{Cone}_{X'\cup W}) \leq R''_{j-1}(R'_j+1)$. For $[v_0,\dots,v_j] \in \overrightarrow{X'}(j)$,

$$|\mathrm{supp}(\mathrm{Cone}_{X'\cup W}(\mathbb{1}_{[v_0,\dots,v_j]}))| = |\mathrm{supp}(\mathrm{Cone}_{X'}(\mathbb{1}_{[v_0,\dots,v_j]}))| \leq R'_j \leq R''_{j-1}(R'_j+1).$$

For $[w,v_1,\dots,v_j] \in \overrightarrow{X'\cup W}(k)$ where $w \in W$,

$$\begin{aligned}
&|\mathrm{supp}(\mathrm{Cone}_{X'\cup W}(\mathbb{1}_{[v_0,\dots,v_j]}))| \\
&\leq |\mathrm{supp}(\mathrm{Cone}_{X'}(\mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\dots,v_j]})))| + |\mathrm{supp}([\mathbb{1}_{[w]},\mathrm{Cone}_{L_w}(\mathbb{1}_{[v_1,\dots,v_j]})])| \\
&\leq R'_j R''_{j-1} + R''_{j-1} = R''_{j-1}(R'_j+1)
\end{aligned}$$

as needed. □

5.5 Constructing cone functions by an induction argument

We want to show coboundary expansion for the opposite complex of a spherical, irreducible building of classical type. We will achieve this by induction over the dimension of the opposite complex, separately for the three different types. In this section, we give a general induction argument that works for an arbitrary class of simplicial complexes satisfying certain conditions. In the next chapter, we will then discuss for each type how a class of complexes containing the opposite complexes satisfies the assumptions of the induction. In each step we will consider both the abelian and the non-abelian case. The idea follows the work of [Abr96], the results appeared in [OV25a] for the abelian and [OV25b] for the non-abelian case.

General induction argument - abelian cones We now describe a way of constructing $\mathbb{Z}$-cone functions for a class of simplicial complexes satisfying certain assumptions. This can be seen as a quantitative version of [Abr96, Lemma 22]. We basically only change Assumption (i) and the conclusion. Note that we are only considering $\mathbb{Z}$-cone functions here, thus we omit the $\mathbb{Z}$ from the notation.

Theorem 5.5.1. Let $\mathbf{C}$ be a class of non-empty simplicial complexes. Assume that for any $n \in \mathbb{N}, n \geq 1$ there exists $\ell_n \in \mathbb{N}$ such that for all $\kappa \in \mathbf{C}$ with $\dim \kappa = n$, there exists a filtration $\kappa_0 \subset \kappa_1 \subset \ldots \subset \kappa_\ell = \kappa$ of κ of length $\ell \leq \ell_n$ such that the following is satisfied, where we set $V_i := \kappa_i(0) \setminus \kappa_{i-1}(0)$:

(i) There is an $(n-1)$-cone function Cone_{κ_0} such that

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_0}) \leq f(n),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function depending only on $\mathbf{C}$.

(ii) κ_i is a full subcomplex of κ for all $0 \leq i \leq \ell$, and for all $1 \leq i \leq \ell$ and all vertices $w_1, w_2 \in V_i$ we have $\{w_1, w_2\} \notin \kappa$.

(iii) $\mathrm{lk}_{\kappa_i}(w) \cap \kappa_{i-1}$ is either in $\mathbf{C}$ or can be written as the join of two elements from $\mathbf{C}$ for any $w \in V_i, i \geq 1$.

Then for every $n \in \mathbb{N}_0$ there exists a constant $\mathcal{R}(n)$ such that for every $\kappa \in \mathbf{C}$ with $\dim \kappa = n$ we have: κ has an $(n-1)$-cone function Cone_κ such that

$$\mathrm{Rad}(\mathrm{Cone}_\kappa) \leq \mathcal{R}(n).$$

We can describe $\mathcal{R}(n)$ recursively. We have $\mathcal{R}(0) = 1$. For $n \geq 1$, assume $\mathcal{R}(k)$ is known for $k < n$. Set

$$S(n) = \max_{a,b \in \mathbb{N}_0 : a+b+1=n} ((a+1)\mathcal{R}(a) + 1)\mathcal{R}(b).$$

Then

$$\mathcal{R}(n) = S(n)^{\ell_n} f(n) + \sum_{j=1}^{\ell_n} S(n)^j.$$

Proof. We will prove the result by induction on the dimension n.

For $n = 0$, since $\kappa \neq \emptyset$ there exists $v \in \kappa(0)$ and thus we can define a (-1)-cone function via $\emptyset \mapsto \mathbb{1}_{[v]}$. Every cone function has (-1)-cone radius $= 1$, thus we take $\mathcal{R}(0) = 1$.

Fix $n \geq 1$ and $\kappa \in \mathbf{C}$ with $\dim(\kappa) = n$ and with filtration $\kappa_0 \subset \ldots \subset \kappa_\ell = \kappa$ as in the requirements. We will prove by induction on $0 \leq i \leq \ell$, that there is an $(n-1)$-cone function Cone_{κ_i} such that

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_i}) \leq \mathcal{R}^{(i)}(n)$$

where $\mathcal{R}^{(i)}(n)$ is given inductively by

$$\mathcal{R}^{(0)}(n) = f(n), \qquad \mathcal{R}^{(i)}(n) = S(n)(\mathcal{R}^{(i-1)}(n) + 1) = S(n)^i f(n) + \sum_{j=1}^{i} S(n)^j.$$

For $i = 0$ this follows from Assumption (i). We proceed by induction on i. Fix $1 \leq i \leq \ell$ and assume there exists a constant $\mathcal{R}^{(i-1)}(n)$ and an $(n-1)$-cone function $\mathrm{Cone}_{\kappa_{i-1}}$ such that $\mathrm{Rad}(\mathrm{Cone}_{\kappa_{i-1}}) \leq \mathcal{R}^{(i-1)}(n)$. We want to apply Theorem 5.4.6 to the following setting, where in the notation of Theorem 5.4.6 we have:

$$X = \kappa, X' = \kappa_{i-1}, \ \mathrm{Cone}_{X'} = \mathrm{Cone}_{\kappa_{i-1}}, \ R'_j = \mathcal{R}^{(i-1)}(n), \ W = V_i = \kappa_i(0) \setminus \kappa_{i-1}(0).$$

We check that the conditions of the theorem are satisfied.

(i) $V_i \cap \kappa_{i-1} = \emptyset$ by definition.

(ii) For every $w_1, w_2 \in V_i : \{w_1, w_2\} \notin \kappa_i \subset \kappa$ by Assumption (ii).

(iii) For every $w \in V_i, \mathrm{lk}_\kappa(w) \cap \kappa_{i-1}$ is a non-empty simplicial complex, since by Assumption (iii) we have $\mathrm{lk}_\kappa(w) \cap \kappa_{i-1} \in \mathbf{C}$ or a join of two complexes in $\mathbf{C}$.

(iv) Let $w \in V_i$ and set $L_w := \mathrm{lk}_\kappa(w) \cap \kappa_{i-1}$. By Assumption (iii), L_w is either in $\mathbf{C}$ or is a join of two elements of $\mathbf{C}$. In the first case we use the induction hypothesis and get an $(n-2)$-cone function Cone_{L_w} such that

$$\mathrm{Rad}(\mathrm{Cone}_{L_w}) \leq \mathcal{R}(n-1).$$

In the second case, we can write $L_w = A * B$ for some $A, B \in \mathbf{C}$. Set $n_A = \dim A, n_B = \dim B$ then $n-1 \geq \dim(L_w) = n_A + n_B + 1$, hence $n_A, n_B < n$. Thus by induction there exists an $(n_A - 1)$-cone function Cone_A with $\mathrm{Rad}(\mathrm{Cone}_A) \leq \mathcal{R}(n_A)$ and similarly for B. Using Proposition 5.3.12 we get an $(n-2)$-cone function Cone_{L_w} such that

$$\mathrm{Rad}(\mathrm{Cone}_{L_w}) \leq ((n_A + 1)\mathcal{R}(n_A) + 1)\mathcal{R}(n_B) \leq S(n).$$

Note that we simplified the bound coming from Proposition 5.3.12 by only considering the largest possible of the cases. We chose $S(n)$ to be the maximum over all the bounds that can appear at this step. Note that in particular $\mathcal{R}(n-1) \leq S(n)$. Thus in the notation of Theorem 5.4.6, we have $R''_j = S(n)$.

Applying Theorem 5.4.6 we get an $(n-1)$-cone function Cone_{κ_i} for $\kappa_i = \kappa_{i-1} \cup V_i$ such that

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_i}) \leq S(n)(\mathcal{R}^{i-1}(n)).$$

Thus we have $\mathcal{R}^{(i)}(n) = S(n)(\mathcal{R}^{(i-1)}(n) + 1)$ and solving the recursion gives the expression of $\mathcal{R}^{(i)}(n)$.

Now, since $\kappa_\ell = \kappa$, we get an $(n-1)$-cone function Cone_κ with $\mathrm{Rad}(\mathrm{Cone}_\kappa) \leq \mathcal{R}^{(\ell)}(n)$. We would like to set $\mathcal{R}(n) = \mathcal{R}^{(\ell)}(n)$. But the bound has to hold for all possible $\kappa \in \mathbf{C}$ of dimension n. Since $\mathcal{R}^{(i)}(n)$ is increasing in i, we take $\mathcal{R}(n) = \mathcal{R}^{(\ell_n)}(n) = S(n)^{\ell_n} f(n) + \sum_{j=1}^{\ell_n} S(n)^j$. □

Note that the κ_i in the filtration do not need to be in the class $\mathbf{C}$ themselves.

Remark 5.5.2. Compared to [Abr96, Lemma 22] we changed Assumption (i) from just asking for κ_0 to be contractible to requiring an explicit cone function, and we removed the dependency on the underlying building, since the theorem also works in this wider generality. In particular, if we want to show that a certain class, for which Abramenko already showed that it satisfies the assumptions of [Abr96, Lemma 22], also satisfies the assumptions of this theorem, we need to check Assumption (i), while Assumptions (ii) and (iii) are usually already covered.

General induction argument - non-abelian Here we state and proof the non-abelian version of Theorem 5.5.1.

Theorem 5.5.3. Let $\mathbf{C}$ be a class of non-empty simplicial complexes. Assume that for any $n \in \mathbb{N}, n \geq 2$ there exists $\ell_n \in \mathbb{N}$ such that for all $\kappa \in \mathbf{C}$ with $\dim \kappa = n$, there exists a filtration $\kappa_0 \subset \kappa_1 \subset \ldots \subset \kappa_\ell = \kappa$ of κ of length $\ell \leq \ell_n$ such that the following is satisfied (set $V_i := \kappa_i(0) \setminus \kappa_{i-1}(0)$):

(i) There is a 1-cone function Cone_{κ_0} such that

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_0}) \leq f(n),$$

where $f : \mathbb{N} \to \mathbb{R}$ is a function depending only on $\mathbf{C}$.

(ii) κ_i is a full subcomplex of κ for all $0 \leq i \leq \ell$ and for all vertices $w_1, w_2 \in V_i$, $1 \leq i \leq \ell$, we have $\{w_1, w_2\} \notin \kappa$.

(iii) $\mathrm{lk}_{\kappa_i}(w) \cap \kappa_{i-1}$ is either in $\mathbf{C}$ or can be written as the join of two elements from $\mathbf{C}$ for any $w \in V_i, 1 \leq i \leq \ell$.

Furthermore, we assume that there exists $\ell_1 \in \mathbb{N}$ such that for any $\kappa \in \mathbf{C}$ with $\dim(\kappa) = 1$, there exists a filtration $\kappa_0 \subset \kappa_1 \subset \ldots \subset \kappa_\ell = \kappa$ of κ of length $\ell \leq \ell_1$ such that the following is satisfied (set $V_i := \kappa_i(0) \setminus \kappa_{i-1}(0)$):

(i) There is a 0-cone function Cone_{κ_0} such that

$$\mathrm{Rad}_0(\mathrm{Cone}_{\kappa_0}) \leq f(1),$$

(ii) κ_i is a full subcomplex of κ and for all vertices $w_1, w_2 \in V_i$ we have $\{w_1, w_2\} \notin \kappa$, for all $0 \leq i \leq \ell$,

(iii) $\mathrm{lk}_{\kappa_i}(w) \cap \kappa_{i-1}$ is non-empty for all $w \in V_i, 1 \leq i \leq \ell$.

Then for every $n \in \mathbb{N}, n \geq 2$ there exists a constant $\mathcal{R}(n)$ such that for every $\kappa \in \mathbf{C}$ with $\dim \kappa = n$ we have: κ has a 1-cone function Cone_κ such that

$$\mathrm{Rad}(\mathrm{Cone}_\kappa) \leq \mathcal{R}(n).$$

Furthermore, every $\kappa \in \mathbf{C}$ with $\dim(\kappa) = 1$ admits a 0-cone function with $\mathrm{Rad}_0(\mathrm{Cone}_\kappa) \leq f(1) + \ell_1$.

We can describe $\mathcal{R}(n)$ recursively. We have $\mathcal{R}(0) = 1$. For $n \geq 1$, assume $\mathcal{R}(k)$ is known for $k < n$. Set

$$S(n) = \max\{2, \mathcal{R}(n-1)\}.$$

Then

$$\mathcal{R}(n) = S(n)^{\ell_n} f(n) + \sum_{j=1}^{\ell_n} S(n)^j.$$

Proof. We will prove the result by induction on the dimension n.

For $n = 1$, we can apply Lemma 5.4.4 inductively to each step of the filtration to get a 0-come function Cone_κ with $\mathrm{Rad}_0(\mathrm{Cone}_\kappa) \leq f(1) + \ell$.

Fix $n \geq 2$ and $\kappa \in \mathbf{C}, \dim(\kappa) = n$ with filtration $\kappa_0 \subset \ldots \subset \kappa_\ell = \kappa$ as in the requirements. We will prove by induction on $0 \leq i \leq \ell$, that there is a 1-cone function Cone_{κ_i} such that

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_i}) \leq \mathcal{R}^{(i)}(n)$$

where $\mathcal{R}^{(i)}(n)$ is given inductively by

$$\mathcal{R}^{(0)}(n) = f(n), \qquad \mathcal{R}^{(i)}(n) = S(n)(\mathcal{R}^{(i-1)}(n) + 1) = S(n)^i f(n) + \sum_{j=1}^{i} S(n)^j.$$

For $i = 0$ this follows from Assumption (i). We proceed by induction on i. Fix $1 \leq i \leq \ell$ and assume there exists a constant $\mathcal{R}^{(i-1)}(n)$ and a 1-cone function $\mathrm{Cone}_{\kappa_{i-1}}$ such that $\mathrm{Rad}(\mathrm{Cone}_{\kappa_{i-1}}) \leq \mathcal{R}^{(i-1)}(n)$. We want to apply Theorem 5.4.3 to the following setting:

$$X = \kappa, X' = \kappa_{i-1},\ \mathrm{Cone}_{X'} = \mathrm{Cone}_{\kappa_{i-1}},\ R' = \mathcal{R}^{(i-1)}(n),\ W = V_i = \kappa_i(0) \setminus \kappa_{i-1}(0).$$

We check that the conditions of the theorem are satisfied.

(i) $W \cap X' = \emptyset$ by definition.

(ii) For every $w_1, w_2 \in W : \{w_1, w_2\} \notin \kappa_i \subset \kappa$ by Assumption (ii).

(iii) For every $w \in W$ set $L_w = \mathrm{lk}_X(w) \cap X'$. Then L_w is a non-empty simplicial complex, since by Assumption (iii) we have $L_w \in \mathbf{C}$ or a join of two complexes in $\mathbf{C}$.

(iv) By Assumption (iii), L_w is either in $\mathbf{C}$ or is a join of two elements of $\mathbf{C}$. In the first case we use the induction hypothesis and get an 0-cone function Cone_{L_w} such that

$$\mathrm{Rad}_0(\mathrm{Cone}_{L_w}) \leq \mathcal{R}(n-1).$$

In the second case, we can write $L_w = A * B$ for some $A, B \in \mathbf{C}$. Thus by Lemma 5.3.3, there exists a 0-cone function Cone_{A*B} with $\mathrm{Rad}_0(\mathrm{Cone}_{A*B}) \leq 2$.

Hence we can set $R'' = \max\{2, \mathcal{R}(n-1)\} = S(n)$.

Applying Theorem 5.4.3 we get a 1-cone function $\mathrm{Cone}_{X' \cup W}$ such that

$$\mathrm{Rad}(\mathrm{Cone}_{X' \cup W}) \leq R''(R'+1)$$

which in the original formulation means

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_i}) \leq S(n)(\mathcal{R}^{(i-1)}(n)+1).$$

Thus we have $\mathcal{R}^{(i)}(n) = S(n)(\mathcal{R}^{(i-1)}(n)+1)$ and solving the recursion gives the second expression of $\mathcal{R}^{(i)}(n)$.

Now, since $\kappa_\ell = \kappa$, we get a 1-cone function Cone_κ with $\mathrm{Rad}(\mathrm{Cone}_\kappa) \leq \mathcal{R}^{(\ell)}(n)$. We would like to set $\mathcal{R}(n) = \mathcal{R}^{(\ell)}(n)$. But the bound has to hold for all possible $\kappa \in \mathbf{C}$ of dimension n. Since $\mathcal{R}^{(i)}(n)$ is increasing in i, we take $\mathcal{R}(n) = \mathcal{R}^{(\ell_n)}(n) = S(n)^{\ell_n} f(n) + \sum_{j=1}^{\ell_n} S(n)^j$.

$\square$

Chapter 6

Cosystolic and coboundary expansion

In this chapter, we show that many KMS complexes give rise to cosystolic expanders and that certain congruence KMS complexes give rise to 2-dimensional coboundary expanders also with non-abelian coefficients.

We want to apply the local-to-global machinery from Theorem 2.3.16 and Theorem 2.3.22. Since we have already shown that the KMS complexes are good local-spectral expanders, it remains to show that all proper links (i.e. links of simplices of dimension between 0 and codimension 2) are coboundary expanders. Due to the dependency of the constants in the local-to-global theorems, we need to bound the coboundary expansion independently of the size of the underlying field (since by increasing the size of the field we can make sure that the local spectral expansion is sufficiently good). By Corollary 4.2.9, we know that every proper link is isomorphic to the opposite complex of a spherical building. If the corresponding building is not irreducible, then it is the join of its irreducible components, and this also holds for the opposite complexes, see [Abr96, Lemma 16]. Thus, we want to show that the opposite complex of an irreducible spherical building is a coboundary expander by constructing a cone function. We achieve this for the opposite complexes of type A_n, C_n and D_n using their geometric description from Section 3.7 (recall that for Coxeter complexes and hence buildings there is no difference between type B_n and type C_n). Using that we can construct cone functions for joins given we have cone functions for the factors, we cover all opposite complexes of classical type, i.e. where each connected component of the Dynkin diagram is of type A_n, C_n or D_n.

We construct the cone functions for the opposite complexes of type A_n and C_n by showing that the classes $\mathbf{C}_A, \mathbf{C}_A \cup \mathbf{C}_C$ from Definitions 3.7.4 and 3.7.8 satisfy the assumptions of the above induction arguments Theorems 5.5.1 and 5.5.3 both in the abelian and non-abelian case. We follow the machinery presented in [Abr96] closely, which itself builds on previous work [Qui73; Vog81; AA93].

We proceed as follows

(i) Each complex κ in one of the classes $\mathbf{C}_A, \mathbf{C}_C$ is the flag complex of a certain set of vector subspaces. We recall the fact from [Abr96] that at least one of these subspaces has dimension 1.

(ii) We use this subspace to define a filtration $\kappa_0 \subset \ldots \subset \kappa_\ell = \kappa$ as in [Abr96].

(iii) We show that κ_0 admits a cone function and we bound its radius.

(iv) We recall from the work in [Abr96] that $\mathrm{lk}_{\kappa_i}(U) \cap \kappa_{i-1}$ for all $U \in \kappa_i(0) \setminus \kappa_{i-1}(0)$ has the desired structure, i.e. is in the class we consider or can be written as a join of two elements in there.

(v) We conclude that the class satisfies the required assumptions and hence get a cone function with bounded radius for each element, in particular for the opposite complexes.

Note that Step (iii) is the only step that we have to consider separately for the abelian and non-abelian case.

The type D_n case is more involved. Here, we do not directly consider opposite complexes in thick D_n buildings but in so called weak C_n buildings since they are easier to describe as flag complexes. Thus we start by showing that a cone function (both abelian and non-abelian) for the weak C_n case gives rise to a cone function in the thick D_n case. We also need an adapted version of the general induction argument. We then proceed as follows.

(i) We define a class $\mathbf{C}_1$.

(ii) We define a filtration $\kappa_0 \subset \ldots \subset \kappa_\ell = \kappa$ for each $\kappa \in \mathbf{C}_1$.

(iii) We show that κ_0 admits a cone function with bounded radius.

(iv) We conclude that $\mathbf{C}_1$ satisfies the assumptions of the adapted general induction argument.

(v) We define a class $\mathbf{C}_2$.

(vi) We proceed as for $\mathbf{C}_1$.

(vii) We define a class $\mathbf{C_3}$.

(viii) We define a filtration $\kappa_0 \subset \ldots \subset \kappa_\ell = \kappa$ for each $\kappa \in \mathbf{C}_3$.

(ix) We show that κ_0 admits a cone function with bounded radius.

(x) We show that $\mathbf{C_1} \cup \mathbf{C_2} \cup \mathbf{C}_3$ satisfies the assumptions of the adapted general induction argument.

(xi) We conclude that all opposite complexes of type D_n admit a cone function.

Note that we again only need to treat those steps both for the abelian and non-abelian case where we construct a cone function for κ_0.

6.1 Cone functions for opposite complexes of type A_n

To show that the complex opposite the fundamental chamber in a spherical building of type A_n has a $\mathbb{Z}$-cone function with bounded cone radius, we show that the class $\mathbf{C}_A$ from Definition 3.7.4 satisfies the assumptions of Theorem 5.5.1. In parallel, we will show that the class $\mathbf{C}_A$ satisfies the assumptions of Theorem 5.5.3.

Fix a $\kappa \in \mathbf{C}_A$ of dimension $n \geq 0$, i.e. a (skew) field K, a vector space V of dimension $n+2$ over K and $\mathcal{E}$ a finite set of subspaces of V satisfying the technical assumption in Definition 3.7.4. Then $\kappa = \operatorname{Flag} X_{\mathcal{E}}(V)$ where $X_{\mathcal{E}}(V) = \{0 \neq U < V \mid U \pitchfork \mathcal{E}\}$. Note that in this case, the corresponding building $\Delta = \operatorname{Flag}\{0 \neq U < V\}$ is of type A_{n+1}.

We recall the first fact that appears in the proof of [Abr96, Proposition 12].

Fact 6.1.1. If $T_{\mathcal{E}}(V) \in \mathbf{C}_A$, then there exists $L \in X_{\mathcal{E}}(V)$ such that L is a line, i.e., $\dim(L) = 1$.

Let $L \in X_{\mathcal{E}}(V)$ be a line. Define

$$Y_0 = \{U \in X_{\mathcal{E}}(V) : L + U \in X_{\mathcal{E}}(V)\},$$

and further define κ_0 to be the subcomplex of $T_{\mathcal{E}}(V)$ spanned by Y_0.

For $1 \leq i \leq n+1$, define $Y_i \subseteq X_{\mathcal{E}}(V)$ as follows:

$$Y_i = \{U \in X_{\mathcal{E}}(V) \mid \dim(U) \geq n+2-i \text{ or } U \in Y_0\}.$$

For every such i, let κ_i be the subcomplex of $T_{\mathcal{E}}(V)$ spanned by Y_i.

Lemma 6.1.2. There is a $(\mathbb{Z}, n-1)$-cone function $\operatorname{Cone}_{\kappa_0}$ such that

$$\operatorname{Rad}(\operatorname{Cone}_{\kappa_0}) \leq n+2.$$

Proof. Let Γ_0 be the subcomplex of κ_0 spanned by $Z_0 = \{U \in X_{\mathcal{E}}(V) : L \leq U\}$. For $1 \leq i \leq n+1$, we denote

$$Z_i = \{U \in Y_0 \mid \dim(U) \leq i \text{ or } U \in Z_0\}.$$

Further denote $\Gamma_i = \operatorname{Flag} Z_i$. Note that $\Gamma_{n+1} = \kappa_0$. We will show by induction that for every $0 \leq i \leq n+1$, there is an $(n-1)$-cone function $\operatorname{Cone}_{\Gamma_i}$ such that $\operatorname{Rad}(\operatorname{Cone}_{\Gamma_i}) \leq i+1$.

For $i = 0$, we note that Γ_0 is the ball of radius 1 around L in κ and can be written as $\Gamma_0 = \{L\} * \operatorname{lk}_\kappa(L)$. By Proposition 5.3.1, there is an $(n-1)$-cone function $\operatorname{Cone}_{\Gamma_0}$ such that $\operatorname{Rad}(\operatorname{Cone}_{\Gamma_0}) \leq 1$.

Let $1 \leq i \leq n+1$ and assume there is an $(n-1)$-cone function $\operatorname{Cone}_{\Gamma_{i-1}}$ such that $\operatorname{Rad}(\operatorname{Cone}_{\Gamma_{i-1}}) \leq i$. We will apply Theorem 5.4.6 in order to bound the cone radius of Γ_i. We note that any two $U_1, U_2 \in Z_i \setminus Z_{i-1}, U_1 \neq U_2$ are of the same dimension and thus are not connected by an edge. Hence Assumption Item (ii) is satisfied.

Let $U \in Z_i \setminus Z_{i-1}$. We will show that $\operatorname{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$ is a join of the vertex $\{U + L\}$ with the complex spanned by all the other vertices in $\operatorname{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$.

Note that since $U \notin Z_0$ we have $U + L \neq U$, and $U + L \in Z_0$. Further, recall that $\mathrm{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$ is a clique complex, thus it is enough to show for every vertex U' in $\mathrm{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$ that is not $L + U$, that there is an edge connecting U' and $L + U$. Let U' be such a vertex.

First, we will deal with the case where $\dim(U') < i$. In that case (using the fact that $\dim(U) = i$), U' being in $\mathrm{lk}_{\kappa_0}(U)$ implies that $U' \leq U$ and thus $U' \leq U + L$, i.e., U' is connected by an edge to $U + L$ as needed.

Second, assume that $\dim(U') > i$. From the definition of Z_{i-1}, it follows that $U' \in Z_0$, i.e., that $L \leq U'$. Also, U' is in $\mathrm{lk}_{\kappa_0}(U)$ and thus $U \leq U'$. It follows that $U + L \leq U'$, i.e., U' is connected by an edge to $U + L$ as needed.

By Proposition 5.3.1, for every $U \in Z_i \setminus Z_{i-1}$, there is an $(n-2)$-cone function $\mathrm{Cone}_{\mathrm{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}}$ such that $\mathrm{Rad}(\mathrm{Cone}_{\mathrm{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}}) \leq 1$.

By Theorem 5.4.6 (with $R''_k = 1$ for every k), it follows that there is an $(n-1)$-cone function Cone_{Γ_i} such that $\mathrm{Rad}(\mathrm{Cone}_{\Gamma_i}) \leq i + 1$ as needed. □

Analogously, we get for non-abelian cone functions the following.

Lemma 6.1.3. There is a NAC Cone_{κ_0} such that

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_0}) \leq n + 2.$$

Proof. We consider the same filtration $\Gamma_0 \subset \ldots \subset \Gamma_{n+1} = \kappa_0$ with $\Gamma_i = \mathrm{Flag}\, Z_i$ as in the proof of Lemma 6.1.2. We know that $\Gamma_0 = \{L\} * \mathrm{lk}_\kappa(L)$. By Proposition 5.3.1, there is NAC Cone_{Γ_0} such that $\mathrm{Rad}(\mathrm{Cone}_{\Gamma_0}) \leq 1$.

Let $1 \leq i \leq n + 1$ and assume there is a NAC $\mathrm{Cone}_{\Gamma_{i-1}}$ such that $\mathrm{Rad}(\mathrm{Cone}_{\Gamma_{i-1}}) \leq i$. From the previous proof, we know that for every $U \in Z_i \setminus Z_{i-1}$ the complex $\mathrm{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$ is a join of the vertex $\{U + L\}$ with the complex spanned by all the other vertices in $\mathrm{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$. Thus, by Proposition 5.3.1, there is 0-cone function $\mathrm{Cone}_{\mathrm{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}}$ such that $\mathrm{Rad}(\mathrm{Cone}_{\mathrm{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}}) \leq 1$.

By Corollary 5.4.5, it follows that there is a NAC Cone_{Γ_i} such that $\mathrm{Rad}(\mathrm{Cone}_{\Gamma_i}) \leq i+1$ as needed. □

The proof of [Abr96, Proposition 12] also shows the following.

Fact 6.1.4. We denote $Y = X_{\mathcal{E}}(V), \kappa = T_{\mathcal{E}}(V)$ and for every $U \in T_{\mathcal{E}}(V)$ we denote the link of U by $\mathrm{lk}_\kappa(U)$. For every $1 \leq i \leq n + 1$ and every $U \in Y_i \setminus Y_{i-1}$, it holds that $\mathrm{lk}_\kappa(U) \cap \kappa_{i-1} = T_{\mathcal{E}'}(U) * T_{\overline{\mathcal{E}}}(V/U)$ such that

- The set $\mathcal{E}'$ consists of subsets of U satisfying $\sum_{j=1}^{n+1} \binom{n}{j-1} e'_j \leq |K|$, hence $T_{\mathcal{E}'}(U) \in \mathbf{C}_A$.
- $\overline{\mathcal{E}}$ is a set of subsets of V/U that satisfy $\sum_{j=1}^{n+1} \binom{n}{j-1} \overline{e}_j \leq |K|$ and hence $T_{\mathcal{E}'}(V/U) \in \mathbf{C}_A$.

Corollary 6.1.5. For every $n \in \mathbb{N} \cup \{0\}$ there is a constant $\mathcal{R}(n)$ such that for every $T_{\mathcal{E}}(V) \in \mathbf{C}_A$ with $\dim T_{\mathcal{E}}(V) = n$ there is a $(\mathbb{Z}, n-1)$-cone function $\mathrm{Cone}_{T_{\mathcal{E}}(V)}$ such that

$$\mathrm{Rad}(\mathrm{Cone}_{T_{\mathcal{E}}(V)}) \leq \mathcal{R}(n).$$

In particular, if Δ is a building of type A_n over a field K with $|K| \geq 2^{n-1}$ and $a \in \Delta$ a simplex, then there exists a $(\mathbb{Z}, n-2)$-cone function $\mathrm{Cone}_{\Delta_{\mathrm{op}}(a)}$ of $\Delta_{\mathrm{op}}(a)$ such that

$$\mathrm{Rad}(\mathrm{Cone}_{\Delta_{\mathrm{op}}(a)}) \leq \mathcal{R}(n-1).$$

Proof. The result follows from Theorem 5.5.1 together with Lemma 6.1.2 and Fact 6.1.4. In the notation of Theorem 5.5.1, we have $f(n) = n + 2$ and $\ell_n = n + 1$. □

Analogously we have the following for non-abelian cones.

Corollary 6.1.6. For every $n \in \mathbb{N}$ there is a constant $\mathcal{R}(n)$ such that for every $T_{\mathcal{E}}(V) \in \mathbf{C}_A$ with $\dim T_{\mathcal{E}}(V) = n$ there is a NAC $\mathrm{Cone}_{T_{\mathcal{E}}(V)}$ such that

$$\mathrm{Rad}(\mathrm{Cone}_{T_{\mathcal{E}}(V)}) \leq \mathcal{R}(n).$$

In particular, if Δ is a building of type A_n $(n \geq 2)$ over a field K with $|K| \geq 2^{n-1}$ and $a \in \Delta$ a simplex, then there exists a NAC $\mathrm{Cone}_{\Delta_{\mathrm{op}}(a)}$ of $\Delta_{\mathrm{op}}(a)$ such that

$$\mathrm{Rad}(\mathrm{Cone}_{\Delta_{\mathrm{op}}(a)}) \leq \mathcal{R}(n-1).$$

Proof. The result follows from Theorem 5.5.3 together with Lemma 6.1.3 and Fact 6.1.4. In the notation of Theorem 5.5.3, we have $f(n) = n + 2$ and $\ell_n = n + 1$. □

6.2 Cone functions for opposite complexes of type C_n

The class $\mathbf{C}$ that will cover the case of opposite complexes in buildings of type C_n will be the union of class $\mathbf{C}_A$ (see Definition 3.7.4) and $\mathbf{C}_C$ (see Definition 3.7.8).

We start by considering $\kappa = T_{\mathcal{E}}(V) = \mathrm{Flag}\, X_{\mathcal{E}}(V) \in \mathbf{C}_C$ of dimension $n \geq 0$ with $V, \mathcal{E}$ as in Definition 3.7.8. The corresponding building is then of type C_{n+1}.

We recall from [Abr96, Proposition 13, Step 1] the following.

Fact 6.2.1. There exists a 1-dimensional subspace $L \in X_{\mathcal{E}}(V)$.

We define the filtration $\kappa_0 \subset \ldots \subset \kappa_{2n+2}$ of κ as follows. We start by setting $X_0 = \{U \in X \mid U \text{ satisfies (i), (ii) or (iii)}\}$ where

(i) $L \leq U$ and $U \pitchfork \mathcal{E}$;

(ii) $L \nleq U \leq L^{\perp}$ and $U \pitchfork \mathcal{E} \cup (\mathcal{E} + L)$;

(iii) $U \nleq L^{\perp}, \dim U > 1$ and $U \pitchfork \mathcal{E} \cup (\mathcal{E} \cap L^{\perp}) \cup (\mathcal{E} + L) \cup ((\mathcal{E} \cap L^{\perp}) + L)$.

Set $\kappa_0 = \operatorname{Flag} X_0$.

Next, we define the rest of the desired filtration in two steps. Set $Z = \{U \in X_{\mathcal{E}}(V) \mid L \leq U \text{ or } U \pitchfork \mathcal{E} + L\}$ and $Y_0 = X_0$. Note that $Y_0 \subseteq Z$. For the first half of the filtration we define

$$Y_i = \{U \in Z \mid U \in Y_0 \text{ or } \dim U \leq i\}, \kappa_i = \operatorname{Flag} Y_i,\ 1 \leq i \leq n+1.$$

For the second half we set

$$Y_i = \{U \in X_{\mathcal{E}}(V) \mid U \in Z \text{ or } \dim U \geq 2n+3-i\}, \kappa_i = \operatorname{Flag} Y_i,\ n \leq i \leq 2n+2.$$

We start by constructing $\mathbb{Z}$ - and non-abelian cone functions for κ_0.

Lemma 6.2.2. With the definitions as above, we have that κ_0 has a $(\mathbb{Z}, n-1)$-cone function $\operatorname{Cone}_{\kappa_0}$ such that

$$\operatorname{Rad}(\operatorname{Cone}_{\kappa_0}) \leq 2n+3.$$

Proof. Let $(T_{\mathcal{E}}(V), \operatorname{Flag} X) \in \mathbf{C}_C$ as above. Thus $\dim \kappa = n$. In particular, any totally isotropic subspace of V has dimension at most $n+1$.

We will apply Theorem 5.4.6 inductively to two different filtrations of κ_0. The first one is defined as follows.

$$\begin{aligned} Y_0 &= \{U \in X \text{ satisfying (i)}\} \\ Y_i &= Y_{i-1} \cup \{U \in X \mid \dim U = i, U \text{ satisfies (ii)}\} \\ &= \{U \in X \mid \dim U \leq i, U \text{ satisfies (ii)}\} \cup Y_0 \text{ for } 1 \leq i \leq n+1. \end{aligned}$$

We set $\Gamma_i = \operatorname{Flag} Y_i$.

The second filtration is given by

$$\begin{aligned} Z_0 &= Y_{n+1} \\ Z_i &= Z_{i-1} \cup \{U \in X \mid \dim U = n+1-(i-1), U \text{ satisfies (iii)}\} \\ &= Z_0 \cup \{U \in X \mid \dim U \geq n+1-(i-1), U \text{ satisfies (iii)}\} \end{aligned}$$

for $1 \leq i \leq n+1$. We set $\zeta_i = \operatorname{Flag} Z_i$. We have that $\Gamma_{n+1} = \zeta_0$ and $\zeta_{n+1} = \kappa_0$.

Next, we want to show that for each Γ_i there exists an $(n-1)$-cone function $\operatorname{Cone}_{\Gamma_i}$ such that $\operatorname{Rad}(\operatorname{Cone}_{\Gamma_i}) \leq i+1$ for all $0 \leq i \leq n+1$. We do this by induction on i.

For $i = 0$ we have $\Gamma_0 = \operatorname{st}_\kappa(L) = \{L\} * \operatorname{lk}_\kappa(L)$. Hence by Proposition 5.3.10 there exists an $(n-1)$-cone function $\operatorname{Cone}_{\Gamma_0}$ with $\operatorname{Rad}(\operatorname{Cone}_{\Gamma_0}) \leq 1$.

For the following step, fix $1 \leq i \leq n+1$. We want to apply Theorem 5.4.6 to the following set-up. Here κ_0 corresponds to the complex called X in Theorem 5.4.6, Γ_{i-1} corresponds to X' , and $W_i := Y_i \setminus Y_{i-1}$ to W i.e. the set of vertices that we want to add. We check that the conditions of Theorem 5.4.6 are satisfied.

For Assumption (i) note that $W_i \cap \Gamma_{i-1} = \emptyset$ by definition. Assumption (ii) holds since all subspaces in W_i have the same dimension, hence they are not connected in κ.

For (iii) and (iv), let $w \in W_i$. Then $\operatorname{lk}_{\kappa_0}(w) = \operatorname{Flag}\{u \in \kappa_0(0) \mid u < w \text{ or } w > u\}$. Furthermore, for $u \in Y_{i-1}$ we have $u + L \in X_0$, since $u \pitchfork \mathcal{E} + L \iff u + L \pitchfork \mathcal{E}$, see [Abr96]. Since $L \leq w + L$, we have $w + L \in Y_0 \subset Y_{i-1}$. Next, we want to show that $\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1}$ is a join of $\{w + L\}$ with the flag complex of all other vertices from $\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1}$. Since $\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1}$ is a clique complex, it suffices to check that every vertex different from $w + L$ is adjacent to $w + L$. Let $u \in (\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1})(0) \setminus \{w + L\}$. Then we differentiate two cases:

- Case 1: $\dim u \leq i - 1$: then $u \leq w \leq w + L$
- Case 2: $\dim u > i$ then $u \in Y_0$, thus $w \leq u$ and $L \leq u$ thus $w + L \leq u$.

Hence, any other vertex is connected to $w + L$ by an edge and thus

$$\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1} = \{w + L\} * \mathrm{Flag}\left((\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1})(0) \setminus \{w + L\}\right).$$

By Proposition 5.3.10 we have $\mathrm{Rad}(\mathrm{Cone}_{\mathrm{lk}_{\kappa_0}(w)\cap\Gamma_{i-1}}) \leq 1$.

Thus we can apply Theorem 5.4.6 and get an $(n-1)$-cone function Cone_{Γ_i} satisfying $\mathrm{Rad}(\mathrm{Cone}_{\Gamma_i}) \leq i + 1$.

We treat the second filtration similarly. We want to show that for every $0 \leq i \leq n+1$ there is an $(n-1)$-cone function Cone_{ζ_i} such that $\mathrm{Rad}(\mathrm{Cone}_{\zeta_i}) \leq n + 2 + i$. We again proceed by induction on i.

For $i = 0$ we have that $\zeta_0 = \Gamma_{n+1}$ has a cone function with radius $\leq n+2$ by the above reasoning.

Now fix $1 \leq i \leq n+1$. We want to use Theorem 5.4.6 with κ_0 corresponding to X, ζ_{i-1} to X', and $Z_i \setminus Z_{i-1} = W$. Then Assumptions (i) and (ii) of Theorem 5.4.6 are satisfied by the same argument as above.

For Assumptions (iii) and (iv), fix $w \in W$ and consider

$$\mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1} = \mathrm{Flag}\,\{u \in Z_{i-1} \mid u < w \text{ or } u > w\}.$$

We show that every vertex (different from $w \cap L^{\perp}$) is connected to $w \cap L^{\perp}$ hence

$$\mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1} = \left\{w \cap L^{\perp}\right\} * \mathrm{Flag}\left\{u \in Z_{i-1} \setminus \left\{w \cap L^{\perp}\right\} \mid u < w \text{ or } u > w\right\}.$$

Let $u \in \mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}(0) = \{u \in Z_{i-1} \mid u < w \text{ or } u > w\}$. If $w \leq u$ then $w \cap L^{\perp} \leq u$ and thus $\{u, w \cap L^{\perp}\} \in \mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}$. If $u \leq w$, then $\dim u < \dim w = n+1-i+1$ and hence $u \in Z_0$. Therefore u satisfies either (i) or (ii). But if it satisfies (i), then $L \leq u \leq w$, a contradiction to $w \in Z_i \setminus Z_{i-1}$. Thus, u satisfies (ii), and in particular $u \leq L^{\perp}$. Therefore, $u \leq w \cap L^{\perp}$ and hence $\{u, w \cap L^{\perp}\} \in \mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}$.

Therefore, by Proposition 5.3.10, $\mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}$ has a cone radius bounded by 1. Applying Theorem 5.4.6 we get a cone function Cone_{ζ_i} satisfying

$$\mathrm{Rad}(\mathrm{Cone}_{\zeta_i}) \leq \mathrm{Rad}(\mathrm{Cone}_{\zeta_{i-1}}) + 1 \leq n + 2 + i.$$

Since $\kappa_0 = \zeta_{n+1}$ we get the desired result. □

We now state and prove the corresponding result for non-abelian cone functions.

Lemma 6.2.3. Let κ_0 be as above. Then it has a NAC Cone_{κ_0} such that

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_0}) \leq 2n+3.$$

Proof. We consider the same filtration $\Gamma_0 \subset \ldots \subset \Gamma_{n+1} = \zeta_0 \subset \ldots \subset \zeta_{n+1} = \kappa_0$ as in the proof of Lemma 6.2.2 above. Now, we want the apply Corollary 5.4.5. We start by showing that for each Γ_i there exists a NAC Cone_{Γ_i} such that $\mathrm{Rad}(\mathrm{Cone}_{\Gamma_i}) \leq i+1$. We do this by induction on i.

For $i = 0$ we have $\Gamma_0 = \mathrm{st}_\kappa(L) = \{L\} * \mathrm{lk}_\kappa(L)$. Hence by Section 5.3 there exists a NAC Cone_{Γ_0} with $\mathrm{Rad}(\mathrm{Cone}_{\Gamma_0}) \leq 1$.

For the following step, fix $1 \leq i \leq n+1$. We want to apply Corollary 5.4.5 to the following set-up. Here κ_0 corresponds to the complex called X in Corollary 5.4.5, Γ_{i-1} corresponds to X' , and $W_i = Y_i \setminus Y_{i-1}$ to W. We check that the conditions of Corollary 5.4.5 are satisfied. Conditions (i) and (ii) hold as in the proof above.

For Condition (iii), let $w \in W_i$. Then, using the results above,

$$\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1} = \{w+L\} * \mathrm{Flag}\left((\mathrm{lk}_{\kappa_0}(w) \cap \Gamma_{i-1})(0) \setminus \{w+L\}\right).$$

By Proposition 5.3.1 we have a 0-cone function $\mathrm{Cone}_{\mathrm{lk}_{\kappa_0}(w)\cap\Gamma_{i-1}}$ with 0-cone radius bounded by 1.

Thus we can apply Corollary 5.4.5 and get a NAC Cone_{Γ_i} satisfying $\mathrm{Rad}(\mathrm{Cone}_{\Gamma_i}) \leq i+1$.

We treat the second filtration similarly. We want to show that for every $0 \leq i \leq n+1$ there is a NAC Cone_{ζ_i} such that $\mathrm{Rad}(\mathrm{Cone}_{\zeta_i}) \leq n+2+i$. We again proceed by induction on i.

For $i = 0$ we have that $\zeta_0 = \Gamma_{n+1}$ has a cone function with radius $\leq n+2$ by the above reasoning.

Now fix $1 \leq i \leq n+1$. We want to use Corollary 5.4.5 with κ_0 corresponding to X, ζ_{i-1} to X', and $Z_i \setminus Z_{i-1} = W$. Then Conditions (i) and (ii) of Corollary 5.4.5 are satisfied by the same argument as above.

For Condition (iii), fix $w \in W$. From the reasoning in the proof of Lemma 6.2.2, we know

$$\mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1} = \left\{w \cap L^\perp\right\} * \mathrm{Flag}\left\{u \in Z_{i-1} \setminus \left\{w \cap L^\perp\right\} \mid u < w \text{ or } u > w\right\}.$$

Thus by Proposition 5.3.1, $\mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}$ has a 0-cone function with cone radius bounded by 1. Applying Corollary 5.4.5 we get a NAC Cone_{ζ_i} satisfying

$$\mathrm{Rad}(\mathrm{Cone}_{\zeta_i}) \leq \mathrm{Rad}(\mathrm{Cone}_{\zeta_{i-1}}) + 1 \leq n+2+i.$$

Since $\kappa_0 = \zeta_{n+1}$ we get the desired result. □

From [Abr96, Propsition 13] we get the following.

Fact 6.2.4. Let $\kappa_0 \subset \ldots \subset \kappa_{2n+2} = \kappa$ be the filtration as above. Then for every $w \in \kappa_i(0) \setminus \kappa_{i-1}(0), 1 \leq i \leq 2n+2$ we have $\mathrm{lk}_{\kappa_i}(w) \cap \kappa_{i-1}$ is in $\mathbf{C} = \mathbf{C}_A \cup \mathbf{C}_C$ or is a join of two elements from $\mathbf{C}$.

Combining the above results, we get:

Corollary 6.2.5. The class **C** satisfies the requirements of Theorem 5.5.1 and Theorem 5.5.3.

Corollary 6.2.6. Let $\Delta = \operatorname{Flag} X(V)$ be a classical C_n building as described in Section 3.7 (note that $\dim \Delta = n-1$). Assume that K is infinite or $K = \mathbb{F}_q$ and

- $q \geq 2^{2n-2}$ if f is alternating and $Q = 0$,
- $q \geq 2^{4n-4}$ if $m = 2n$ and $\sigma \neq \mathrm{id}$,
- $q \geq 2^{2n-1}$ if $m = 2n+1$ or $2n+2$.

In particular, if Δ is the building coming from the BN-pair of a Chevalley group of type B_n or C_n over the field $\mathbb{F}_q$ we require $q \geq 2^{2n-1}$.

Then for any $a \in \Delta$ there exists a $(\mathbb{Z}, n-2)$-cone function $\operatorname{Cone}_{\Delta_{\mathrm{op}}(a)}$ of $\Delta_{\mathrm{op}}(a)$ with

$$\operatorname{Rad}(\operatorname{Cone}_{\Delta_{\mathrm{op}}(a)}) \leq \mathcal{R}(n-1)$$

where $\mathcal{R}(n-1)$ is as in Theorem 5.5.1 with $f(n) = 2n+1$ and $\ell_n = 2n+2$.

Furthermore, for any $a \in \Delta$ there exists a NAC $\operatorname{Cone}_{\Delta_{\mathrm{op}}(a)}$ of $\Delta_{\mathrm{op}}(a)$ with

$$\operatorname{Rad}(\operatorname{Cone}_{\Delta_{\mathrm{op}}(a)}) \leq \mathcal{R}(n-1)$$

where $\mathcal{R}(n-1)$ is as in Theorem 5.5.3 with $f(n) = 2n+1$ and $\ell_n = 2n+2$.

6.3 Cone functions for opposite complexes of type D_n

The proof that opposition complexes of buildings of type D_n are coboundary expanders is the most involved case. We use the notation introduced in Section 3.7. We follow the procedure described at the beginning of the chapter.

Cone functions for a subdivision We will later apply an adapted version of the general induction argument to a class of complexes containing those of the form $T_{\mathcal{E}}(V) = \operatorname{Flag} X_{\mathcal{E}}(V)$ as in Definition 3.7.9. But by Proposition 3.7.10, the opposite complexes in a building of type D_n are of the form $\tilde{T}_{\mathcal{E}}(V)$. Thus we start by showing that a cone function (both abelian and non-abelian) for $T_{\mathcal{E}}(V)$ also gives rise to a cone function for $\tilde{T}_{\mathcal{E}}(V)$. This can be seen as a quantitative version of [Abr96, Corollary 17 (ii)]. Before we start with the case of $\mathbb{Z}$-cone functions, we recall some notation.

Let (V, Q, f) be a pseudo-quadratic space over a field K, such that V is $m = 2n$-dimensional and Q, f satisfy certain properties (see Section 3.7). We set

$$\begin{aligned} X &= \{0 < U < V \mid U \text{ is totally isotropic}\}, & \Delta &= \operatorname{Flag} X, \\ \tilde{X} &= \{U \in X \mid \dim X \neq n-1\}, & \tilde{\Delta} &= \operatorname{Orifl}(\tilde{X}), \\ X_{\mathcal{E}}(V) &= \{U \in X \mid U \tilde{\pitchfork} \mathcal{E}\}, & T_{\mathcal{E}}(V) &= \operatorname{Flag} X_{\mathcal{E}}(V), \\ \tilde{X}_{\mathcal{E}}(V) &= \{U \in \tilde{X} \mid U \tilde{\pitchfork} \mathcal{E}\}, & \tilde{T}_{\mathcal{E}}(V) &= \operatorname{Orifl} \tilde{X}_{\mathcal{E}}(V). \end{aligned}$$

Proposition 6.3.1. Let $\tau = \{E_1 < \cdots < E_r\} \in \tilde{\Delta}$ and set $\mathcal{E}(\tau) = \{E_i, E_i^\perp \mid 1 \leq i \leq r\}$. If $T_{\mathcal{E}(\tau)}(V)$ has an $(n-1)$-cone function $\mathrm{Cone}_{T_{\mathcal{E}(\tau)}(V)}$ with $\mathrm{Rad}(\mathrm{Cone}_{T_{\mathcal{E}(\tau)}(V)}) \leq c$ for some $c \geq 0$ then there exists an $(n-1)$-cone function $\mathrm{Cone}_{\tilde{T}_{\mathcal{E}(\tau)}(V)}$ with $\mathrm{Rad}(\mathrm{Cone}_{\tilde{T}_{\mathcal{E}(\tau)}(V)}) \leq 2c$.

Proof. For shorter notation, we write $T = T_{\mathcal{E}(\tau)}(V), Y = X_{\mathcal{E}(\tau)}(V)$ and $\tilde{T} = \tilde{T}_{\mathcal{E}(\tau)}(V), \tilde{Y} = \tilde{X}_{\mathcal{E}(\tau)}(V)$. We will also write $C_j(T)$ for $C_j(T, \mathbb{Z})$ since we only deal with integer coefficients in this proof.

The idea of the proof is to construct an $(n-1)$-cone function for $\tilde{T}$ given a cone function

$$\mathrm{Cone}_T^j : C_j(T) \to C_{j+1}(T), -1 \leq j \leq n-1$$

by defining maps $f_j : C_j(\tilde{T}) \to C_j(T)$ for $-1 \leq j \leq n-1$ and $g_j : C_j(T) \to C_j(\tilde{T})$ for $0 \leq j \leq n$ such that $g_{j+1} \circ \mathrm{Cone}_T^j \circ f_j$ gives rise to a cone function of $\tilde{T}$.

As already observed in [Abr96], T is a subdivision of $\tilde{T}$ in the following way. Note that $\tilde{Y} \subset Y$ and $Y \setminus \tilde{Y} = \{U \in Y \mid \dim U = n-1\}$. If two subspaces $U, W \in \tilde{Y}$ are connected by an edge in T then the same is true in $\tilde{T}$, but we have additional edges in $\tilde{T}$, namely in the case when $\dim U = \dim W = n$ and $\dim(U \cap W) = n-1$. To go from $\tilde{T}$ to T, we subdivide the edge $\{U, W\}$ by adding the vertex $U \cap W$ and connecting U, W and every vertex $U' \in \tilde{Y}$ for which $\{U', U, W\} \in \tilde{T}$ to $U \cap W$.

Recall from the discussion in Section 3.7 that there are two different types of n-dimensional spaces in $\tilde{X}$ and that spaces from the same type cannot have an intersection of dimension $n-1$. Furthermore, each $U \in X$ of dimension $n-1$ is contained in exactly two totally isotropic spaces of dimension n. If $U \in Y$ of dimension $n-1$, i.e. $U \pitchfork \mathcal{E}$, with $W_1, W_2 \in X$ of dimension n such that $U = W_1 \cap W_2$ then by [Abr96, Lemma 31 (iii)] we have that $W_i \pitchfork \mathcal{E}$ and hence $W_1, W_2 \in Y$.

In particular, any simplex in $\tilde{T}$ contains at most two vertices which are spaces of dimension n.

For each $U \in Y \setminus \tilde{Y}$ pick one of the two n-dimensional subspaces containing it and denote it by W_U. Note that if $W \in Y$ is connected to U by an edge in T, i.e. $W \subseteq U$ or $U \subseteq W$ then $\{W_U, W\} \in \tilde{T}(1)$ if and only if $W \neq W_U$.

We define the following two subsets of $T(j)$ for $0 \leq j \leq n$:

$$I_j = \{\sigma \in T(j) \mid \exists U \in \sigma(0) : \dim U = n-1\} = T(j) \setminus \widetilde{T}(j),$$
$$J_j = \{\sigma \in T(j) \mid \exists U \in \sigma(0) : \dim U = n-1 \text{ and } W_U \notin \sigma(0)\}.$$

Let $\sigma \in J_j$, then we define $\sigma\langle U, W_U\rangle \in \overrightarrow{\tilde{T}(j)}$ to be the oriented simplex with vertex W_U at the position of U instead of U. This is well defined by the above observation. Similarly, if $\sigma \in \overrightarrow{\tilde{T}}(j)$ with $W_1, W_2 \in \sigma(0), \dim W_1 = \dim W_2 = n$ and $U = W_1 \cap W_2$ has dimension $n-1$, we write $\sigma\langle W_i, U\rangle$ for the oriented simplex where we replaced W_i with U. This is now a well-defined simplex in $\overrightarrow{T(j)}$.

Next, we define the function $f_j : C_j(\tilde{T}) \to C_j(T), 0 \leq j \leq n-1$ by its action on chains of the form $\mathbb{1}_\sigma$ and then extending $\mathbb{Z}$-linearly:

$$f_j(\mathbb{1}_\sigma) = \begin{cases} \mathbb{1}_\sigma & \text{if } \sigma \in \overrightarrow{T(j)} \cap \overrightarrow{\tilde{T}(j)} \\ \mathbb{1}_{\sigma\langle W_1,U\rangle} + \mathbb{1}_{\sigma\langle W_2,U\rangle} & \text{if } \sigma \in \overrightarrow{\tilde{T}(j)} \setminus \overrightarrow{T(j)}, W_1, W_2 \in \sigma(0), \dim W_i = n, \\ & \qquad U = W_1 \cap W_2, \dim U = n-1. \end{cases}$$

On the other hand, we define

$$\begin{aligned} g_j : C_j(T) &\to C_j(\tilde{T}) \\ \mathbb{1}_\sigma &\mapsto \begin{cases} \mathbb{1}_\sigma & \text{if } \sigma \in \overrightarrow{T(j)} \cap \overrightarrow{\tilde{T}(j)} \\ \mathbb{1}_{\sigma\langle U,W_U\rangle} & \text{if } \sigma \in J_j \\ 0 & \text{if } \sigma \in I_j \setminus J_j. \end{cases} \end{aligned}$$

We define $\mathrm{Cone}^j_{\tilde{T}} = g_{j+1} \circ \mathrm{Cone}^j_T \circ f_j, -1 \leq j \leq n-1$, and $\mathrm{Cone}_{\tilde{T}} = \oplus_{j=-1}^{n-1} \mathrm{Cone}^j_{\tilde{T}}$. The majority of the rest of the proof is dedicated to showing that this is indeed a cone function. As a concatenation of $\mathbb{Z}$-linear functions it is $\mathbb{Z}$-linear. If $[U]$ is the apex of Cone_T then the apex of $\mathrm{Cone}_{\tilde{T}}$ is again $[U]$ if $\dim U \neq n-1$ and otherwise it is $[W_U]$.

Hence, we are left with showing the cone equation. Let $\sigma \in \overrightarrow{T(j)}$. We write $\mathrm{Cone}_T(\mathbb{1}_\sigma) = \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau \mathbb{1}_\tau$ for $\lambda_\tau \in \mathbb{Z}$. In the following, if $\tau \in I_{j+1}$ is a simplex, then U will always denote the subspace of dimension $n-1$ in $\tau(0)$. Let $[\sigma : \tau]$ denote the oriented incidence number as defined in Section 2.2. Then

$$\begin{aligned} \partial_{j+1}(g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_\sigma))) &= \partial_{j+1}(\mathrm{Cone}_T(\mathbb{1}_\sigma)) - \sum_{\tau \in I_{j+1}} \lambda_\tau \partial_{j+1} \mathbb{1}_\tau + \sum_{\tau \in J_{j+1}} \lambda_\tau \partial_{j+1} \mathbb{1}_{\tau\langle U,W_U\rangle} \\ &= \mathbb{1}_\sigma - \mathrm{Cone}_T(\partial_j \mathbb{1}_\sigma) - \sum_{\tau \in I_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)}} [\tau : \gamma] \mathbb{1}_\gamma \\ &\quad + \sum_{\tau \in J_{j+1}} \lambda_\tau [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}} + \sum_{\tau \in J_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_{\gamma\langle U,W_U\rangle} \end{aligned}$$

where we used that

$$\partial_{j+1} \mathbb{1}_{\tau\langle U,W_U\rangle} = \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_{\gamma\langle U,W_U\rangle} + [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}}.$$

Note that $(\tau\langle U, W_U\rangle) \setminus \{W_U\} = \tau \setminus \{U\}$ and $[\tau : \gamma\langle U, W_U\rangle] = [\tau : \gamma]$.

We take a closer look at the second half of that expression:

$$
\begin{aligned}
&- \sum_{\tau \in I_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)}} [\tau : \gamma] \mathbb{1}_\gamma + \sum_{\tau \in J_{j+1}} \lambda_\tau [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}} \\
&\quad + \sum_{\tau \in J_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_{\gamma \langle U, W_U \rangle} \\
= &- \sum_{\tau \in I_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_\gamma + \sum_{\tau \in J_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_{\gamma \langle U, W_U \rangle} \\
&\quad - \sum_{\tau \in I_{j+1}} \lambda_\tau [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}} + \sum_{\tau \in J_{j+1}} \lambda_\tau [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}} \\
= &- \sum_{\tau \in I_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_\gamma + \sum_{\tau \in J_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_{\gamma \langle U, W_U \rangle} \\
&\quad - \sum_{\tau \in I_{j+1} \setminus J_{j+1}} \lambda_\tau [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}}.
\end{aligned}
$$

On the other hand, let $\lambda'_\gamma \in \mathbb{Z}$ such that $\operatorname{Cone}_T(\partial \mathbb{1}_\sigma) = \sum_{\gamma \in \overrightarrow{T(j)}} \lambda'_\gamma \mathbb{1}_\gamma$. Then

$$
g_j(\operatorname{Cone}_T(\partial \mathbb{1}_\sigma)) = \operatorname{Cone}_T(\partial \mathbb{1}_\sigma) - \sum_{\gamma \in I_j} \lambda'_\gamma \mathbb{1}_\gamma + \sum_{\gamma \in J_j} \lambda'_\gamma \mathbb{1}_{\gamma \langle U, W_U \rangle}.
$$

What are the λ'_γ in terms of the λ_τ?

$$
\begin{aligned}
\operatorname{Cone}_T(\partial \mathbb{1}_\sigma) &= \mathbb{1}_\sigma - \partial_{j+1} \operatorname{Cone}_T(\mathbb{1}_\sigma) = \mathbb{1}_\sigma - \partial_{j+1} \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau \mathbb{1}_\tau \\
&= \mathbb{1}_\sigma - \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)}} [\tau : \gamma] \mathbb{1}_\gamma \\
&= \mathbb{1}_\sigma - \sum_{\gamma \in \overrightarrow{T(j)}} \left(\sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_\gamma.
\end{aligned}
$$

Hence if $\gamma \neq \sigma$ then

$$
\lambda'_\gamma = - \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma]
$$

and otherwise

$$
\lambda'_\sigma = 1 - \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \sigma].
$$

To compare the expression of $g_j(\operatorname{Cone}_T(\partial_j \mathbb{1}_\sigma))$ with $\partial_{j+1}(g_{j+1}(\operatorname{Cone}_T(\mathbb{1}_\sigma)))$ we furthermore need the following observation comparing the sets I_j with I_{j+1} and J_j with J_{j+1}.

$$
\left\{ (\tau, \gamma) : \gamma \in I_j, \tau \in \overrightarrow{T(j+1)}, \gamma \subset \tau \right\} = \left\{ (\tau, \gamma) : \tau \in I_{j+1}, \gamma \in \overrightarrow{T(j)}, \gamma \subset \tau, \gamma \neq \tau \setminus \{U\} \right\}
$$

and

$$\begin{aligned}
&\left\{(\tau,\gamma) : \gamma \in J_j, \tau \in \overrightarrow{T(j+1)}, \gamma \subset \tau\right\} \\
=&\left\{(\tau,\gamma) : \tau \in J_{j+1}, \gamma \in \overrightarrow{T(j)}, \gamma \subset \tau, \gamma \neq \tau \setminus \{U\}\right\} \cup \{(\tau,\gamma) : \tau \in I_{j+1} \setminus J_{j+1}, \gamma = \tau \setminus \{W_U\}\}.
\end{aligned}$$

We use the above result to further investigate the second half of the expression for $g_j(\mathrm{Cone}_T(\partial \mathbb{1}_\sigma))$. We first assume that $\sigma \notin I_j$.

$$\begin{aligned}
&-\sum_{\gamma\in I_j} \lambda'_\gamma \mathbb{1}_\gamma + \sum_{\gamma \in J_j} \lambda'_\gamma \mathbb{1}_{\gamma\langle U,W_U\rangle} \\
&= -\sum_{\gamma\in I_j}\left(-\sum_{\tau\in\overrightarrow{T(j+1)}} \lambda_\tau[\tau:\gamma]\right)\mathbb{1}_\gamma + \sum_{\gamma\in J_j}\left(-\sum_{\tau\in\overrightarrow{T(j+1)}} \lambda_\tau[\tau:\gamma]\right)\mathbb{1}_{\gamma\langle U,W_U\rangle} \\
&= \sum_{\tau\in I_{j+1}} \sum_{\gamma\in\overrightarrow{T(j)}:\gamma\neq\tau\setminus\{U\}} [\tau:\gamma]\lambda_\tau \mathbb{1}_\gamma - \sum_{\tau\in J_{j+1}} \sum_{\gamma\in\overrightarrow{T(j)}:\gamma\neq\tau\setminus\{U\}} [\tau:\gamma]\lambda_\tau \mathbb{1}_{\gamma\langle U,W_U\rangle} \\
&\quad - \sum_{\tau\in I_{j+1}\setminus J_{j+1}} [\tau:\tau\setminus\{W_U\}]\lambda_\tau \mathbb{1}_{(\tau\setminus\{W_U\})\langle U,W_U\rangle}.
\end{aligned}$$

As second case, assume $\sigma \in I_j \setminus J_j$.

$$\begin{aligned}
&g_j(\mathrm{Cone}_T(\partial\mathbb{1}_\sigma)) = \mathrm{Cone}_T(\partial\mathbb{1}_\sigma) - \sum_{\gamma\in I_j\setminus\{\sigma\}} \lambda'_\gamma\mathbb{1}_\gamma + \sum_{\gamma\in J_j}\lambda'_\gamma \mathbb{1}_{\gamma\langle U,W_U\rangle} - \lambda'_\sigma\mathbb{1}_\sigma \\
&= \mathrm{Cone}_T(\partial\mathbb{1}_\sigma) - \sum_{\gamma\in I_j\setminus\{\sigma\}}\left(-\sum_{\tau\in\overrightarrow{T(j+1)}}\lambda_\tau[\tau:\gamma]\right)\mathbb{1}_\gamma + \sum_{\gamma\in J_j}\left(-\sum_{\tau\in\overrightarrow{T(j+1)}}\lambda_\tau[\tau:\gamma]\right)\mathbb{1}_{\gamma\langle U,W_U\rangle} \\
&\quad -\left(1-\sum_{\tau\in\overrightarrow{T(j+1)}}\lambda_\tau[\tau:\sigma]\right)\mathbb{1}_\sigma \\
&= \mathrm{Cone}_T(\partial\mathbb{1}_\sigma) - \sum_{\gamma\in I_j}\left(-\sum_{\tau\in\overrightarrow{T(j+1)}}\lambda_\tau[\tau:\gamma]\right)\mathbb{1}_\gamma + \sum_{\gamma\in J_j}\left(-\sum_{\tau\in\overrightarrow{T(j+1)}}\lambda_\tau[\tau:\gamma]\right)\mathbb{1}_{\gamma\langle U,W_U\rangle} - \mathbb{1}_\sigma.
\end{aligned}$$

As third and last case, we consider $\sigma \in J_j \subset I_j$.

$$
\begin{aligned}
&g_j(\mathrm{Cone}_T(\partial \mathbb{1}_\sigma)) = \mathrm{Cone}_T(\partial \mathbb{1}_\sigma) - \sum_{\gamma \in I_j \setminus \{\sigma\}} \lambda'_\gamma \mathbb{1}_\gamma + \sum_{\gamma \in J_j \setminus \{\sigma\}} \lambda'_\gamma \mathbb{1}_{\gamma \langle U, W_U \rangle} - \lambda'_\sigma \mathbb{1}_\sigma + \lambda'_\sigma \mathbb{1}_{\sigma \langle U, W_U \rangle} \\
&= \mathrm{Cone}_T(\partial \mathbb{1}_\sigma) - \sum_{\gamma \in I_j \setminus \{\sigma\}} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_\gamma + \sum_{\gamma \in J_j \setminus \{\sigma\}} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_{\gamma \langle U, W_U \rangle} \\
&\quad - \left(1 - \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \sigma] \right) \mathbb{1}_\sigma + \left(1 - \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \sigma] \right) \mathbb{1}_{\sigma \langle U, W_U \rangle} \\
&= \mathrm{Cone}_T(\partial \mathbb{1}_\sigma) - \sum_{\gamma \in I_j} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_\gamma + \sum_{\gamma \in J_j} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_{\gamma \langle U, W_U \rangle} \\
&\quad - \mathbb{1}_\sigma + \mathbb{1}_{\sigma \langle U, W_U \rangle}.
\end{aligned}
$$

We now look at almost the complete cone equation for $\mathrm{Cone}_{\tilde{T}}$, still not considering f_j. Let $\sigma \in \overrightarrow{T(j)} \cap \overrightarrow{\tilde{T}(j)}$, then we get

$$
\begin{aligned}
&\partial_{j+1} g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_\sigma)) + g_j(\mathrm{Cone}_T(\partial_j \mathbb{1}_\sigma)) = \mathbb{1}_\sigma \underbrace{- \mathrm{Cone}_T(\partial_j \mathbb{1}_\sigma)}_{(1)} \\
&\underbrace{- \sum_{\tau \in I_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_\gamma}_{(2)} + \underbrace{\sum_{\tau \in J_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_{\gamma \langle U, W_U \rangle}}_{(3)} \\
&\underbrace{- \sum_{\tau \in I_{j+1} \setminus J_{j+1}} \lambda_\tau [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}}}_{(4)} \\
&\underbrace{+ \mathrm{Cone}_T(\partial \mathbb{1}_\sigma)}_{(1)} + \underbrace{\sum_{\tau \in I_{j+1}} \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \lambda_\tau \mathbb{1}_\gamma}_{(2)} \underbrace{- \sum_{\tau \in J_{j+1}} \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \lambda_\tau \mathbb{1}_{\gamma \langle U, W_U \rangle}}_{(3)} \\
&\underbrace{- \sum_{\tau \in I_{j+1} \setminus J_{j+1}} [\tau : \tau \setminus \{W_U\}] \lambda_\tau \mathbb{1}_{(\tau \setminus \{W_U\}) \langle U, W_U \rangle}}_{(4)}.
\end{aligned}
$$

The parts marked (1),(2), and (3) cancel each other out respectively. It remains to show that the same is true for the parts marked (4). Note that the sums in the parts marked (4) run over the same elements. Hence we need to show that for any $\tau \in I_{j+1} \setminus J_{j+1}$ we have

$$[\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}} = -[\tau : \tau \setminus \{W_U\}] \mathbb{1}_{(\tau \setminus \{W_U\}) \langle U, W_U \rangle}$$

Note that as an unoriented simplex $\tau \setminus \{U\}$ is the same as $(\tau \setminus \{W_U\})\langle U, W_U\rangle$ (in both cases we have all the vertices of τ except U). Let $\tau = [U_0, \ldots, U_{j+1}]$ and assume that $U = U_\ell$ and $W_U = U_k$ for some $0 \leq \ell, k \leq j+1$. Then $[\tau : \tau \setminus \{U\}] = (-1)^\ell$ and $[\tau : \tau \setminus \{W_U\}] = (-1)^k$. Further note that we can obtain$(\tau \setminus \{W_U\})\langle U, W_U\rangle$ from $\tau \setminus \{U\}$ by cyclically permuting the vertices of index between $\min\{\ell, k\}$ and $\max\{\ell, k\} - 1$. The cycle has length $|k - \ell| - 1$ and thus signature $(-1)^{k-\ell-1}$. Hence $(-1)^k \mathbb{1}_{(\tau\setminus\{W_U\})\langle U, W_U\rangle} = (-1)^k \cdot (-1)^{k-\ell-1} \mathbb{1}_{\tau\setminus\{U\}} = -(-1)^\ell \mathbb{1}_{\tau\setminus\{U\}}$ which is what we wanted to show.

To summarize, we get

$$\partial_{j+1} g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_\sigma)) + g_j(\mathrm{Cone}_T(\partial_j \mathbb{1}_\sigma)) = \mathbb{1}_\sigma.$$

Analogously, we get for $\sigma \in I_j \setminus J_j$

$$\partial_{j+1} g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_\sigma)) + g_j(\mathrm{Cone}_T(\partial_j \mathbb{1}_\sigma)) = 0$$

and for $\sigma \in J_j$

$$\partial_{j+1} g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_\sigma)) + g_j(\mathrm{Cone}_T(\partial_j \mathbb{1}_\sigma)) = \mathbb{1}_{\sigma\langle U, W_U\rangle}.$$

Note that if $\sigma \in \overrightarrow{T(j)} \cap \overrightarrow{\tilde{T}(j)}$, then $f_j(\mathbb{1}_\sigma) = \mathbb{1}_\sigma$ which shows that the cone equation holds in this case. To conclude the proof, we need to check the cone equation also for the case where $\sigma \in \overrightarrow{\tilde{T}(j)} \setminus \overrightarrow{T(j)}$. In this case $\sigma(0)$ contains two subspaces W_1, W_2 with $\dim W_i = n$ and $U := W_1 \cap W_2$ has dimension $n-1$. Without loss of generality, assume that $W_U = W_1$. We get $f_j(\mathbb{1}_\sigma) = \mathbb{1}_{\sigma\langle W_1, U\rangle} + \mathbb{1}_{\sigma\langle W_2, U\rangle}$. We have that $\sigma\langle W_1, U\rangle \in J_j$ and $\sigma\langle W_2, U\rangle \in I_j \setminus J_j$ thus we can apply the above reasoning to both of them separately and get, using linearity:

$$\begin{aligned}
&\partial_{j+1} g_{j+1}(\mathrm{Cone}_T(f_j(\mathbb{1}_\sigma))) + g_j(\mathrm{Cone}_T(f_j(\partial_j \mathbb{1}_\sigma)))\\
&= \partial_{j+1} g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_{\sigma\langle W_1, U\rangle})) + g_j(\mathrm{Cone}_T(\partial_j \mathbb{1}_{\sigma\langle W_1, U\rangle}))\\
&\quad + \partial_{j+1} g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_{\sigma\langle W_2, U\rangle})) + g_j(\mathrm{Cone}_T(\partial_j \mathbb{1}_{\sigma\langle W_2, U\rangle}))\\
&= \mathbb{1}_{(\sigma\langle W_1, U\rangle)\langle U, W_U\rangle} + 0 = \mathbb{1}_\sigma.
\end{aligned}$$

For the last equality, we used that $(\sigma\langle W_1, U\rangle)\langle U, W_U\rangle = \sigma$ since we first replace W_1 in σ by U and by applying $\langle U, W_U\rangle$ we replace U by $W_U = W_1$ and end up where we started.

For bounding the cone radius of $\mathrm{Cone}_{\tilde{T}}$ note that f_j at most doubles the size of the support of a chain, while applying g_j will not increase the size of the support of a chain. Hence in the worst case, i.e. if $\sigma \in I_j$ then

$$\begin{aligned}
|\mathrm{supp}(\mathrm{Cone}_{\tilde{T}}(\mathbb{1}_\sigma))| &\leq |\mathrm{supp}(\mathrm{Cone}_T(\mathbb{1}_{\sigma\langle W_1, U\rangle}))| + |\mathrm{supp}(\mathrm{Cone}_{\tilde{T}}(\mathbb{1}_{\sigma\langle W_2, U\rangle}))|\\
&\leq 2\,\mathrm{Rad}_j\,\mathrm{Cone}_T \leq 2c.
\end{aligned}$$

Hence we get the desired bound on the cone radius. □

Next, we show the analogous result for non-abelian cones.

Proposition 6.3.2. Let $\tau = \{E_1 < \cdots < E_r\} \in \tilde{\Delta}$ and set $\mathcal{E}(\tau) = \{E_i, E_i^{\perp} \mid 1 \leq i \leq r\}$. If $T_{\mathcal{E}(\tau)}(V)$ has an 1-cone function $\mathrm{Cone}_{T_{\mathcal{E}(\tau)}(V)}$ with $\mathrm{Rad}(\mathrm{Cone}_{T_{\mathcal{E}(\tau)}(V)}) \leq c$ for some $c \geq 0$ then there exists a 1-cone function $\mathrm{Cone}_{\tilde{T}_{\mathcal{E}(\tau)}(V)}$ with $\mathrm{Rad}_0(\mathrm{Cone}_{\tilde{T}_{\mathcal{E}(\tau)}(V)}) \leq c$ and $\mathrm{Rad}_1(\mathrm{Cone}_{\tilde{T}_{\mathcal{E}(\tau)}(V)}) \leq 2c$.

To prove the proposition, we need the following definition and auxiliary lemmas.

Definition 6.3.3. Let $\tau = \{E_1 < \cdots < E_r\} \in \tilde{\Delta}$ and set $\mathcal{E}(\tau) = \{E_i, E_i^{\perp} \mid 1 \leq i \leq r\}$. Denote $T = T_{\mathcal{E}(\tau)}(V)$ and $\tilde{T} = \tilde{T}_{\mathcal{E}(\tau)}(V)$. Each $u \in X$ with $\dim u = n-1$ is contained in two n-dimensional spaces in X. We denote them by W_u and $\bar{W}_u$.

We define the following function:

$$\varphi : T_{\mathrm{ord}}(1) \to \text{ paths in } \tilde{T}$$
$$(u,v) \mapsto \begin{cases} (W_u; v) & \text{if } \dim u = n-1, v \neq W_u \\ (W_u) & \text{if } \dim u = n-1, v = W_u \\ (u; W_v) & \text{if } \dim v = n-1, u \neq W_v \\ (W_v) & \text{if } \dim v = n-1, u = W_v \\ (u; v) & \text{else.} \end{cases}$$

Note that this definition is well defined, since if (u,v) is an edge in T and $\dim v = n-1$ then (u, W_v) is an edge in $\tilde{T}$ (either $u \subseteq v \subseteq W_v$ or $u \cap W_v = v$).

Definition/Lemma 6.3.4. The map φ preserves paths, i.e. given a path $P = (v_0; \ldots; v_m)$ in T, we have that $\varphi(P) := \varphi(v_0, v_1) \circ \varphi(v_1, v_2) \circ \ldots \circ \varphi(v_{m-1}, v_m)$ is a path in $\tilde{T}$.

Proof. Given two edges $(u,v), (v,w) \in T_{\mathrm{ord}}(1)$ such that the terminal vertex of one edge equals the initial vertex of the other. Then the terminal vertex of $\varphi((u,v))$ will be equal to the initial vertex of $\varphi((v,w))$ (if $\dim v \neq n-1$ it will be v, otherwise it will be W_v). Hence, the concatenation of the images of the edges of a path is well defined and gives again a path. $\square$

Note that it follows from the definition that $\varphi(P \circ Q) = \varphi(P) \circ \varphi(Q)$ and $\varphi(P^{-1}) = \varphi(P)^{-1}$.

Lemma 6.3.5. Given two paths P, P' in T, if $P \overset{(BT)}{\sim} P'$ then $\varphi(P) \overset{(BT)}{\sim} \varphi(P')$ or $\varphi(P) = \varphi(P')$.

Proof. Since $P \overset{(BT)}{\sim} P'$ there exit paths Q_0, Q_1 and vertices u, v in T such that $P = Q_0 \circ (u; v; u) \circ Q_1^{-1}$ and $P' = Q_0 \circ (u) \circ Q_1^{-1}$. Thus, $\varphi(P) = \varphi(Q_0) \circ \varphi((u; v; u)) \circ \varphi(Q_1)^{-1}$ and similarly for $\varphi(P')$. Now, if $\dim u, \dim v \neq n-1$ then $\varphi((u; v; u)) = (u; v; u)$ and $\varphi((u)) = (u)$ which proves the statement in this case.

If $\dim u = n-1$ then $\varphi((u)) = (W_u)$ and $\varphi((u; v; u)) = (W_u; v; W_u)$ or $\varphi((u; v; u)) = (W_u)$ and the statement follows.

If $\dim v = n-1$ then $\varphi((u)) = (u)$ and $\varphi((u; v; u)) = (u; W_v; u)$ or $\varphi((u; v; u)) = (W_v) = (u)$ (the second case occurs exactly when $u = W_v$). $\square$

u	v	w	$\varphi((u;v;w))$	$\varphi((u;w))$	$\varphi(P)$ vs $\varphi(P')$
$\dim u \neq n-1$	$\dim v \neq n-1$	$\dim w \neq n-1$	$(u;v;w)$	$(u;w)$	(TR)
$u \neq W_w$	$v \neq W_w$	$\dim w = n-1$	$(u;v;W_w)$	$(u;W_w)$	(TR)
$u \neq W_w$	$v = W_w$	$\dim w = n-1$	$(u;W_w)$	$(u;W_w)$	$=$
$u = W_w$	$v \neq W_w$	$\dim w = n-1$	$(u = W_w;v;W_w)$	(W_w)	(BT)
$u \neq W_v$	$\dim v = n-1$	$w \neq W_v$	$(u;W_v;w)$	$(u;w)$	(TR)
$u \neq W_v$	$\dim v = n-1$	$w = W_v$	$(u;W_v)$	$(u;w = W_v)$	$=$
$u = W_v$	$\dim v = n-1$	$w \neq W_v$	$(W_v;w)$	$(u = W_v;w)$	$=$
$\dim u = n-1$	$v \neq W_u$	$w \neq W_u$	$(W_u;v;w)$	$(W_u;w)$	(TR)
$\dim u = n-1$	$v = W_u$	$w \neq W_u$	$(W_u;w)$	$(W_u;w)$	$=$
$\dim u = n-1$	$v \neq W_u$	$w = W_u$	$(W_u;v;w = W_u)$	(W_u)	(BT)

Table 6.1: Relation ship between $\varphi(P)$ and $\varphi(P')$ for each case.

Lemma 6.3.6. Given two paths P, P' in T, if $P \overset{(TR)}{\sim} P'$ then we have $\varphi(P) \overset{(TR)}{\sim} \varphi(P')$, $\varphi(P) \overset{(BT)}{\sim} \varphi(P')$ or $\varphi(P) = \varphi(P')$.

Proof. Since $P \overset{(TR)}{\sim} P'$ there exit paths Q_0, Q_1 and a triangle (u, v, w) in T such that $P = Q_0 \circ (u;v;w) \circ Q_1^{-1}$ and $P' = Q_0 \circ (u;w) \circ Q_1^{-1}$. Thus, $\varphi(P) = \varphi(Q_0) \circ \varphi((u;v;w)) \circ \varphi(Q_1)^{-1}$ and similarly for $\varphi(P')$.

We will proceed by case distinction which we display in Table 6.1. The last column shows the relationship between $\varphi(P)$ and $\varphi(P')$. □

Proof of Proposition 6.3.2. We consider the setting of Definition 6.3.3. Assume that we have a 1-cone function $(v_0, (P_v)_{v \in T(0)}, (T_{(u,v)})_{(u,v) \in T_{\mathrm{ord}}}(1))$ of T. The goal is to construct a 1-cone function $(\tilde{v_0}, (\tilde{P}_v)_{v \in \tilde{T}(0)}, (\tilde{T}_{(u,v)})_{(u,v) \in \tilde{T}_{\mathrm{ord}}(1)})$ for $\tilde{T}$.

For the apex, we set $\tilde{v_0} = v_0$ if $\dim v_0 \neq n-1$ and otherwise $\tilde{v_0} = W_{v_0}$.

Given a vertex $v \in \tilde{T}(0) = \tilde{Y} := X_{\mathcal{E}(\tau)}(V)$, we set $\tilde{P}_v = \varphi(P_v)$. Since $v \in \tilde{Y}$, we have $\dim v \neq n-1$ and hence $\tilde{P}_v$ is a path from $\tilde{v_0}$ to v of length at most the length of P_v.

Let $(u, v) \in \tilde{T}_{\mathrm{ord}}(1)$. To define the contraction $\tilde{T}_{(u,v)}$, we distinguish two cases. First assume that $(u, v) \in T \cap \tilde{T}$, i.e. at least one of the two vertices has dimension less than n. Hence we can consider $T_{(u,v)} = (P_0, \ldots, P_m)$ and define $\tilde{T}_{(u,v)} = (\varphi(P_0), \ldots, \varphi(P_m))$. The auxiliary Lemmas 6.3.4, 6.3.5, 6.3.6 imply that this is a well defined contraction and $|\tilde{T}_{(u,v)}| \leq |T_{(u,v)}|$.

For the second case, let $(u, v) \in \tilde{T}_{\mathrm{ord}}(1)$ with $\dim u = \dim v = n$. Consider $T_{(u,u\cap v)} = (P_0^1, \ldots, P_m^1)$ and $T_{(u\cap v,v)} = (P_0^2, \ldots, P_\ell^2)$. Note that $\varphi((u;u\cap v;v)) = (u;v)$. We set

$$\tilde{P}_0 = \tilde{P}_u \circ \varphi((u, u\cap v) \circ (u\cap v, v)) \circ \tilde{P}_v^{-1} = \varphi(P_u \circ (u;u\cap v;v) \circ P_v^{-1}).$$

By $|P_{u\cap v}| =: s$ many backtracking steps, we can transform $\tilde{P}_0$ to

$$\tilde{P}_s = \varphi(P_u \circ (u, u\cap v) \circ P_{u\cap v}^{-1} \circ P_{u\cap v} \circ (u\cap v, v) \circ P_v^{-1}) = \varphi(P_0^1 \circ P_0^2).$$

Applying the contraction steps from $T_{(u,u\cap v)}$ on the first half of the path and then the steps from $T_{(u\cap v,v)}$ on the second half we get the desired contraction, (again using Lemma 6.3.5 and Lemma 6.3.6). In particular $|\tilde{T}_{(u,v)}|\leq |T_{(u,u\cap v)}|+|T_{(u\cap v,v)}|$. This also proves the bound on the 1-cone radius.

□

Remark 6.3.7. The above reasoning also shows that if T has a 0-cone function, then we can construct a 0-cone function for $\tilde{T}$. Further note that the dimensions of T and $\tilde{T}$ coincide. Hence, we can also get a more general statement of Proposition 6.3.2 saying that given a NAC for T, we can construct a NAC for $\tilde{T}$ and $\operatorname{Rad}(\tilde{T}) \leq 2\operatorname{Rad}(T)$.

Adapted version of the general induction argument For the D_n case, we need the following more specialized version of Theorems 5.5.1 and 5.5.3. In order to state it, we need the following definition.

Definition 6.3.8. Let K be a field, V a K-vector space, Y a collection of subspaces of V. Let $X = \operatorname{Flag} Y$ and $U \in Y$. Then we define

$$Y^{<U} := \{W \in Y \mid W < U\},\ X^{<U} := \operatorname{Flag} Y^{<U}$$
$$Y^{>U} := \{W \in Y \mid W > U\},\ X^{>U} := \operatorname{Flag} Y^{>U}.$$

Remark 6.3.9. Note that

$$\operatorname{lk}_X(U) = X^{<U} * X^{>U}.$$

We start with the result for $\mathbb{Z}$-cone functions.

Theorem 6.3.10. Let $\mathbf{C}$ be a class of tuples $(\kappa,(V,Q,f))$ where (V,Q,f) is a thick pseudo-quadratic space and $\kappa = \operatorname{Flag} Y$ where $Y \neq \emptyset$ is set of vector subspaces of V. Let $n = \dim \kappa$, assume there exists $\ell_n \in \mathbb{N}$ and a filtration $\kappa_0 \subset \kappa_1 \subset \ldots \subset \kappa_\ell = \kappa, \ell \leq \ell_n$ satisfying:

(i) There exists an $(n-1)$-cone function $\operatorname{Cone}_{\kappa_0}$ such that

$$\operatorname{Rad}(\operatorname{Cone}_{\kappa_0}) \leq f(n)$$

where $f : \mathbb{R} \to \mathbb{R}$ is a function depending only on $\mathbf{C}$.

(ii) Every κ_i is a full subcomplex of κ, i.e. $\kappa_i = \operatorname{Flag} Y_i$ for some $Y_i \subseteq Y$ and all vector subspaces in $Y_i \setminus Y_{i-1}$ have the same dimension.

(iii) For every $U \in Y_i \setminus Y_{i-1}$ we have that $\operatorname{lk}_{\kappa_i}(U) \cap \kappa_{i-1} = \kappa_{i-1}^{<U} * \kappa_{i-1}^{>U}$ is such that $\kappa_{i-1}^{<U}, \kappa_{i-1}^{>U} \in \mathbf{C}$ or one (or both) have again a filtration $R_0 \subset \ldots \subset R_h, h \leq \ell_{\dim R_0}$ which starts with an element of $\mathbf{C}$ (i.e. $R_0 \in \mathbf{C}$), such that Condition (ii) is satisfied and is such that for every $W \in R_j(0) \setminus R_{j-1}(0)$ we have $R_{j-1}^{<W}, R_{j-1}^{>W} \in \mathbf{C}$.

Then there exist constants $\mathcal{R}(n)$ depending only on $\mathbf{C}$ and n, and there exists an $(n-1)$-cone function $\operatorname{Cone}_\kappa$ such that

$$\operatorname{Rad}(\operatorname{Cone}_\kappa) \leq \mathcal{R}(n).$$

Proof. The proof is analogous to the one of Theorem 5.5.1. The main difference is that to get a cone function for $\mathrm{lk}_{\kappa_i}(U) \cap \kappa_{i-1}$ it might happen that we cannot just use the induction hypothesis, but that we need to do another induction on the filtration of $\kappa_{i-1}^{<U}$ or $\kappa_{i-1}^{>U}$, using Theorem 5.4.6 in the step of the induction. This gives an even weaker and more complicated bound on the cone radius of the final cone function, thus we decided to not keep track of it. But it will still only depend on n and the class $\mathbf{C}$, not on the field K. □

Theorem 6.3.11. Let $\mathbf{C}$ be a class of tuples $(\kappa, (V, Q, f))$ where (V, Q, f) is a thick pseudo-quadratic space and $\kappa = \mathrm{Flag}\, Y$ where $Y \neq \emptyset$ is a set of vector subspaces of V. Assume that for any $n \in \mathbb{N}, n \geq 2$ there exist $\ell_n \in \mathbb{N}$ such that for all $\kappa \in \mathbf{C}, \dim \kappa = n$ there exists a filtration $\kappa_0 \subset \kappa_1 \subset \ldots \subset \kappa_\ell = \kappa, \ell \leq \ell_n$ satisfying:

(i) There exists an 1-cone function Cone_{κ_0} such that

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_0}) \leq f(n),$$

where $f : \mathbb{N} \to \mathbb{R}$ is a function depending only on $\mathbf{C}$.

(ii) Every κ_i is a full subcomplex of κ, i.e. $\kappa_i = \mathrm{Flag}\, Y_i$ for some $Y_i \subseteq Y$ and all vector subspaces in $Y_i \setminus Y_{i-1}$ have the same dimension.

(iii) For every $U \in Y_i \setminus Y_{i-1}$ we have that $\mathrm{lk}_{\kappa_i}(U) \cap \kappa_{i-1} = \kappa_{i-1}^{<U} * \kappa_{i-1}^{>U}$ is such that $\kappa_{i-1}^{<U}, \kappa_{i-1}^{>U} \in \mathbf{C}$ or one (or both) have again a filtration $R_0 \subset \ldots R_h, h \leq \ell_{\dim R_0}$ which starts with an element of $\mathbf{C}$ (i.e. $R_0 \in \mathbf{C}$), such that Condition (ii) is satisfied and is such that for every $W \in R_j(0) \setminus R_{j-1}(0)$ we have $R_{j-1}^{<W}, R_{j-1}^{>W} \in \mathbf{C}$.

Furthermore, we assume that there exists $\ell_1 \in \mathbb{N}$ such that for any $\kappa \in \mathbf{C}$ with $\dim(\kappa) = 1$, there exists a filtration $\kappa_0 \subset \kappa_1 \subset \ldots \subset \kappa_\ell = \kappa$ of κ of length $\ell \leq \ell_1$ such that the following is satisfied (set $V_i := \kappa_i(0) \setminus \kappa_{i-1}(0)$):

(i) There is a 0-cone function Cone_{κ_0} such that

$$\mathrm{Rad}_0(\mathrm{Cone}_{\kappa_0}) \leq f(1),$$

(ii) κ_i is a full subcomplex of κ and for all vertices $w_1, w_2 \in V_i$ we have $\{w_1, w_2\} \notin \kappa$, for all $0 \leq i \leq \ell$.

(iii) $\mathrm{lk}_{\kappa_i}(w) \cap \kappa_{i-1}$ is non-empty for all $w \in V_i, 1 \leq i \leq \ell$.

Then for every $n \in \mathbb{N}, n \geq 2$ there exists a constant $\mathcal{R}(n)$ such that for every $\kappa \in \mathbf{C}$ with $\dim \kappa = n$ we have: κ has a 1-cone function Cone_κ such that

$$\mathrm{Rad}(\mathrm{Cone}_\kappa) \leq \mathcal{R}(n).$$

Furthermore, every $\kappa \in \mathbf{C}$ with $\dim(\kappa) = 1$ admits a 0-cone function with $\mathrm{Rad}_0(\mathrm{Cone}_\kappa) \leq f(1) + \ell_1$.

Proof. The proof is analogous to the one of Theorem 5.5.3, keeping in mind the same changes as in the abelian case. □

Defining the correct class and showing that it satisfies the assumptions We now describe the set up for the rest of this section. We follow [Abr96, Section 7] closely and also refer to that book for more details. We will define three subclasses $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ and first show that $\mathbf{C}_1, \mathbf{C}_2$ satisfy the assumptions of Theorems 6.3.10 and 6.3.11 to then conclude that $\mathbf{C}_D = \mathbf{C}_1 \cup \mathbf{C}_2 \cup \mathbf{C}_3$ satisfies them as well.

We fix n to be the Witt index of (V, Q, f) hence $\dim V = 2n$ and the corresponding building is $n-1$-dimensional. Recall $X = \{0 < U < V \mid U \text{ totally isotropic}\}$.

Definition 6.3.12 ([Abr96, Definition 12])**.** Let $U, E < V$, $\dim U < n, \dim E = n$. Assume that E is not totally isotropic. Then

$$U@E :\iff U^\perp \cap E \text{ is not totally isotropic.}$$

Set

$$\begin{aligned}\mathcal{U}_n = \mathcal{U}_n(V) &= \{A < V \mid \dim A = n, A \notin X\}\\ \mathcal{M}(A) &= \{M < A \mid \dim M = n-1, M \in X\} \text{ for any } A \in \mathcal{U}_n.\end{aligned}$$

Definition 6.3.13. We define the class $\mathbf{C}_1$ to consist of complexes $\operatorname{Flag} X_{\mathcal{E};\mathcal{F}}(U;V)$ where

- (V, Q, f) is thick pseudo-quadratic space of Witt index n, in particular $\dim V = 2n$;
- $U \in X$ with $\dim U = k \geq 2$;
- $\mathcal{E}$ a finite set of subspaces of U, set $e_j = |\mathcal{E}_j|, 1 \leq j \leq k-1$ like before;
- Let $\mathcal{F} = \mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots \supset \mathcal{F}_{k-1}$ be finite subsets of $\mathcal{U}_n(V)$ satisfying
  - (i) $U \cap F \in \mathcal{E} \cup \{0\}$ for all $F \in \mathcal{F}$;
  - (ii) $\mathcal{F}_j^\perp = \mathcal{F}_j$;
  - (iii) $\dim(U \cap F) \leq k-1-j$ for all $F \in \mathcal{F}_j, 1 \leq j \leq k-1$.
- Assume that $|K| \geq \sum_{j=1}^{k-1} \binom{k-2}{j-1} e_j + 2^{k-1}s$, where $s = |\mathcal{F}|$.

Set

$$X_{\mathcal{E};\mathcal{F}}(U;V) := \{0 < W < U \mid W \pitchfork_U \mathcal{E} \text{ and } W@_V\mathcal{F}_j \text{ for } \dim W = j\}.$$

Fact 6.3.14. Let $\kappa = \operatorname{Flag} X_{\mathcal{E};\mathcal{F}}(U;V) \in \mathbf{C}_1$ with $k = \dim U \geq 2$, in particular κ has dimension $k-2$. Then there exists a 1-dimensional subspace $L \in X_{\mathcal{E};\mathcal{F}}(U;V)$.

We define the filtration for κ as above in the following way. Note that by the above fact $\kappa \neq \emptyset$, thus in particular if $\dim \kappa = 0$ then we get a (-1)-cone function.

Recall $k = \dim U = \dim \kappa + 2$ and assume $k \geq 3$. We set $Y = X_{\mathcal{E};\mathcal{F}}(U;V)$ and define

$$Y_0 := \{A \in Y \mid A + L \in Y\}, \ \kappa_0 = \operatorname{Flag} Y_0.$$

The filtration that we will use consists of two dimension decreasing filtrations. First we define

$$\begin{aligned}
Z &:= \{W \in Y \mid L \leq W \text{ or } W \pitchfork_U (\mathcal{F} \cap U) + L\} \supseteq Y_0 \\
Y_i &:= \{W \in Z \mid W \in Y_0 \text{ or } \dim W \geq k - i\}, \\
\kappa_i &:= \operatorname{Flag} Y_i,\ 1 \leq i \leq k-1.
\end{aligned}$$

The second part of the filtration of κ is defined as:

$$\begin{aligned}
Z_i &:= \{W \in Y \mid W \in Z \text{ or } \dim W \geq k - i\}, \\
\kappa_{k-1+i} &:= \operatorname{Flag} Z_i, 0 \leq i \leq k-1.
\end{aligned}$$

Lemma 6.3.15. With the above setting, $\kappa_0 = \operatorname{Flag} Y_0$ has a $(\mathbb{Z}, k-3)$-cone function Cone_{κ_0} with

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_0}) \leq k.$$

Proof. We proceed similar to the proof of Lemma 6.1.2. Set $B_0 := \{A \in Y \mid L \leq A\}, \beta_0 = \operatorname{Flag} B_0$. Then $\beta_0 = \{L\} * \mathrm{lk}_Y(\{L\})$ and hence it has a $(k-3)$-cone function with cone radius bounded by 1. Next, set $B_i = \{A \in Y_0 \mid L \leq A \text{ or } \dim A \leq i\}, \beta_i = \operatorname{Flag} B_i$ for $1 \leq i \leq k-1$. Then all subspaces in $B_i \setminus B_{i-1}$ have dimension i. Let $A \in B_i \setminus B_{i-1}$. Then

$$\mathrm{lk}_{\beta_i}(A) \cap \beta_{i-1} = \{A + L\} * \operatorname{Flag}\{W \in B_{i-1} \mid B \neq A + L, A < W \text{ or } W < A\}$$

by the same argument as in Lemma 6.1.2. Applying Theorem 5.4.6 inductively, we get a $(k-3)$-cone function $\mathrm{Cone}_{\beta_{k-1}} = \mathrm{Cone}_{\kappa_0}$ with

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_0}) \leq k.$$

□

We get the analogous result for non-abelian cones.

Lemma 6.3.16. With the setting as above, $\kappa_0 = \operatorname{Flag} Y_0$ has a NAC Cone_{κ_0} with radius at most k.

Proof. Set $B_0 := \{A \in Y \mid L \leq A\}, \beta_0 = \operatorname{Flag} B_0$. Then $\beta_0 = \{L\} * \mathrm{lk}_Y(\{L\})$ and hence it has a NAC with cone radius bounded by 1. Next, set $B_i = \{A \in Y_0 \mid L \leq A \text{ or } \dim A \leq i\}, \beta_i = \operatorname{Flag} B_i$ for $1 \leq i \leq k-1$. Then all subspaces in $B_i \setminus B_{i-1}$ have dimension i. Let $A \in B_i \setminus B_{i-1}$. Then $\mathrm{lk}_{\beta_i}(A) \cap \beta_{i-1} = \{A + L\} * \operatorname{Flag}\{W \in B_{i-1} \mid B \neq A + L, A < W \text{ or } W < A\}$ by the same argument as in Lemma 6.1.3. Applying Corollary 5.4.5 inductively, we get a NAC $\mathrm{Cone}_{\beta_{k-1}} = \mathrm{Cone}_{\kappa_0}$ with

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_0}) \leq k.$$

□

We now show, following the arguments in the proof of [Abr96, Lemma 33], that the filtration of κ has the desired structure on the links in each step.

Lemma 6.3.17. Let κ with filtration $\kappa_0 \subset \ldots \subset \kappa_{2k-2} = \kappa$ as above. For every $U \in \kappa_i(0) \setminus \kappa_{i-1}$ we have that

$$\mathrm{lk}_{\kappa_i}(U) \cap \kappa_{i-1} = \kappa_{i-1}^{<U} * \kappa_{i-1}^{>U}$$

is such that $\kappa_{i-1}^{<U}, \kappa_{i-1}^{>U} \in \mathbf{C}_1$ or one (or both) have again a filtration $R_0 \subset \ldots \subset R_h, h \leq \ell_{\dim R_0}$ which starts with an element of $\mathbf{C}_1$ (i.e. $R_0 \in \mathbf{C}_1$), satisfies itself Condition (ii) of Theorem 6.3.10, and is such that for every $W \in R_j(0) \setminus R_{j-1}(0)$ we have $R_{j-1}^{<W}, R_{j-1}^{>W} \in \mathbf{C}_1$.

Proof. We start with the first part of the filtration. Recall

$$\begin{aligned} Z &:= \{W \in Y \mid L \leq W \text{ or } W \pitchfork_U (\mathcal{F} \cap U) + L\} \supseteq Y_0 \\ Y_i &:= \{W \in Z \mid W \in Y_0 \text{ or } \dim W \geq k - i\}, \\ \kappa_i &:= \operatorname{Flag} Y_i, \ 1 \leq i \leq k-1. \end{aligned}$$

This clearly satisfies Condition (ii) in Theorems 6.3.10 and 6.3.11. For Condition (iii), let $W \in Y_i \setminus Y_{i-1}$. Then $Y_{i-1}^{<W} = Y_0^{<W}$ by looking at the dimensions. Step 3 in the proof of [Abr96, Lemma 33] shows that $Y_0^{<W} = X_{\mathcal{E}';\mathcal{F}'}(W;V)$ for some $\mathcal{E}', \mathcal{F}'$. Thus $\kappa_{i-1}^{<W} \in \mathbf{C}_1$. On the other hand, $Y_{i-1}^{>W} = Z^{>W}$. Step 5 in [Abr96, Lemma 33] shows that while $\operatorname{Flag} Z^{>W}$ is not necessary in $\mathbf{C}_1$ it admits itself a filtration of length i satisfying the necessary properties.

Recall that the second part of the filtration has the form

$$\begin{aligned} Z_i &:= \{W \in Y \mid W \in Z \text{ or } \dim W \geq k - i\}, \\ \kappa_{k-1+i} &:= \operatorname{Flag} Z_i, 0 \leq i \leq k-1. \end{aligned}$$

This part of the filtration clearly satisfies Condition (ii). For Condition (iii), let $W \in Z_i \setminus Z_{i-1}$. Then $Z_{i-1}^{>W} = Y^{>W}$ and $\operatorname{Flag} Y^{>W} \in \mathbf{C}_1$ by Step 4 in [Abr96, Lemma 33]. On the other hand, $Z_{i-1}^{<W} = Z^{<W}$ for which we again have $\operatorname{Flag} Z^{<W} \in \mathbf{C}_1$ by [Abr96]. $\square$

Thus we can conclude for the class $\mathbf{C_1}$:

Corollary 6.3.18. The class $\mathbf{C}_1$ satisfies the assumptions of Theorems 6.3.10 and 6.3.11.

We now define the second out of three classes.

Definition 6.3.19. We define the class $\mathbf{C}_2$ to consist of complexes $\operatorname{Flag} Z_{\mathcal{E};\mathcal{F}}(U;V)$ where

- (V, Q, f) is a thick pseudo-quadratic space of Witt index n, in particular $\dim V = 2n$;
- $U \in X = \{0 < W < V \mid W \text{ totally isotropic}\}$ with $\dim U = k \geq 2$;
- $\mathcal{E}$ a finite set of subspaces of U, set $e_j = |\mathcal{E}_j|, 1 \leq j \leq k-1$ like before;
- $\mathcal{F} \subset \mathcal{U}_n(V)$ with $|\mathcal{F}| = s < \infty$ such that
  - (i) $U \cap \mathcal{F} \subseteq \mathcal{E} \cup \{0\}$,
  - (ii) $U \cap F^\perp = 0$ and $\dim(U \cap F) \leq 1$ for all $F \in \mathcal{F}$.

- Assume $|K| \geq \sum_{j=1}^{k-1} \binom{k-2}{j-1} e_j + 2s$.
- Set
$$Z_{\mathcal{E};\mathcal{F}}(U;V) := \{0 < W < U \mid W \pitchfork_U \mathcal{E} \text{ and } W @_V \mathcal{F}\}.$$

We get a similar fact as usual, but this time we are interested in the existence of a hyperplane instead of a line.

Fact 6.3.20. Let $\kappa = \operatorname{Flag} Z_{\mathcal{E};\mathcal{F}}(U;V) \in \mathbf{C}_2$, $\dim U = k \geq 2$, in particular $\dim \kappa = k-2$. Set $Y = Z_{\mathcal{E};\mathcal{F}}(U;V)$. Then there exists $H < U$ with $\dim H = k-1$ and $H \in Y$.

We use this fact to define a filtration for $\kappa = \operatorname{Flag} Z_{\mathcal{E};\mathcal{F}}(U;V) \in \mathbf{C}_2$. For $k = 2$ we have that $\dim \kappa = 0$ hence the desired filtration and cone function exist trivially. Fix $k \geq 3$ and assume $\dim \kappa = k-2$. We set

$$Y_0 := \{B \in Y \mid B \cap H \in Y\}, \kappa_0 = \operatorname{Flag} Y_0$$

and

$$\begin{aligned} Y_i &:= \{W \in Y \mid W \in Y_0 \text{ or } \dim W \leq i\}, \\ \kappa_i &:= \operatorname{Flag} Y_i,\ 1 \leq i \leq k-1. \end{aligned}$$

Note that $\kappa_{k-1} = \kappa$.

We construct a $\mathbb{Z}$-cone function for κ_0.

Lemma 6.3.21. In the above setting, $\kappa_0 = \operatorname{Flag} Y_0$ has a $(\mathbb{Z}, k-3)$-cone function $\operatorname{Cone}_{\kappa_0}$ with
$$\operatorname{Rad}(\operatorname{Cone}_{\kappa_0}) \leq k.$$

Proof. The proof is similar to the proofs of Lemma 6.1.2 and Lemma 6.3.15 , the difference being that we now look at the dual version, where we fix a hyperplane instead of a line.

Set $B_0 = \{A \in Y \mid A \leq H\}, \beta_0 = \operatorname{Flag} B_0$. Then $\beta_0 = \{H\} * \operatorname{lk}_{Y_0}(\{H\})$ hence it has a $(k-3)$-cone function with cone radius bounded by 1.

Next, set $B_i = \{A \in Y_0 \mid A \leq H \text{ or } \dim A \geq k-i\}, \beta_i = \operatorname{Flag} B_i$ for $1 \leq i \leq k-1$. Then the subspaces in $B_i \setminus B_{i-1}$ are all of dimension $k-i$. Let $A \in B_i \setminus B_{i-1}$, then

$$\operatorname{lk}_{\beta_i}(A) \cap \beta_{i-1} = \{A \cap H\} * \operatorname{Flag}\{W \in B_{i-1} \mid W \neq A \cap H, W < A \text{ or } A < W\}.$$

Hence it has a $(k-4)$-cone function with cone radius bounded by 1. Thus we can apply Theorem 5.4.6 inductively and obtain a $(k-3)$-cone function $\operatorname{Cone}_{\beta_{k-1}} = \operatorname{Cone}_{\kappa_0}$ with

$$\operatorname{Rad}(\operatorname{Cone}_{\kappa_0}) \leq k.$$

□

Next, we state the analogous result for non-abelian cones.

Lemma 6.3.22. With the definitions as above, $\kappa_0 = \operatorname{Flag} Y_0$ is $(k-2)$-dimensional and has a NAC Cone_{κ_0} with

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_0}) \leq k.$$

Proof. We consider the filtration $\beta_0 \subset \ldots \subset \beta_{k-1} = \kappa_0$ with $\beta_i = \operatorname{Flag} B_i$ from the proof of Lemma 6.3.21. From the above reasoning, we know $\beta_0 = \{H\} * \mathrm{lk}_{Y_0}(\{H\})$ hence it has a NAC with cone radius bounded by 1.

Let $A \in B_i \setminus B_{i-1}$, then $\mathrm{lk}_{\beta_i}(A) \cap \beta_{i-1} = \{A \cap H\} * \operatorname{Flag}\{W \in B_{i-1} \mid W \neq A \cap H, W < A \text{ or } A < W\}$. Hence it has a 0-cone function with cone radius bounded by 1. Thus we can apply Corollary 5.4.5 inductively and obtain a NAC $\mathrm{Cone}_{B_{k-1}} = \mathrm{Cone}_{\kappa_0}$ with

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_0}) \leq k.$$

□

Lemma 6.3.23. Let κ with filtration $\kappa_0 \subset \ldots \subset \kappa_{n+k-1} = \kappa$ as above. For every $U \in \kappa_i(0) \setminus \kappa_{i-1}$ we have that

$$\mathrm{lk}_{\kappa_i}(U) \cap \kappa_{i-1} = \kappa_{i-1}^{<U} * \kappa_{i-1}^{>U}$$

is such that $\kappa_{i-1}^{<U}, \kappa_{i-1}^{>U} \in \mathbf{C}_2$ or one (or both) have again a filtration $R_0 \subset \ldots \subset R_h, h \leq \ell_{\dim R_0}$ which starts with an element of $\mathbf{C}_2$ (i.e. $R_0 \in \mathbf{C}_2$), satisfies itself Condition (ii) of Theorem 6.3.10, and is such that for every $W \in R_j(0) \setminus R_{j-1}(0)$ we have $R_{j-1}^{<W}, R_{j-1}^{>W} \in \mathbf{C}_2$.

The proof of this lemma follows closely the steps and arguments in the proof of [Abr96, Lemma 34], adapting it to our setting where necessary.

Proof. Recall that the filtration is defined as

$$\begin{aligned} Y_0 &:= \{B \in Y \mid B \cap H \in Y\} \\ Y_i &:= \{W \in Y \mid W \in Y_0 \text{ or } \dim W \leq i\}, \\ \kappa_i &:= \operatorname{Flag} Y_i,\ 0 \leq i \leq k-1. \end{aligned}$$

This clearly satisfies Condition (ii) of Theorems 6.3.10 and 6.3.11. To see that it satisfies Condition (iii), let $W \in Y_i \setminus Y_{i-1}$. Then $Y_{i-1}^{<W} = Y^{<W} = \{0 < A < W \mid A \pitchfork_W \mathcal{E} \cap W\} = Z_{\mathcal{E} \cap W; \emptyset}(W; V)$ and hence $\operatorname{Flag} Y^{<W} \in \mathbf{C}_2$. On the other hand, $Y_{i-1}^{>W} = Y_0^{>W}$ can again be described in such a way that $\operatorname{Flag} Y_0^{>W} \in \mathbf{C}_2$, see Step (5) in [Abr96, Lemma 34]. Since $\kappa_{k-1} = \kappa$, the filtration is indeed as desired. □

Thus we conclude the following.

Corollary 6.3.24. The class $\mathbf{C}_2$ satisfies the assumptions of Theorems 6.3.10 and 6.3.11.

Next, we will define a third class, which will be the one containing the relevant opposite complexes, see also Remark 6.3.31.

Definition 6.3.25. We define the class $\mathbf{C}_3$ to consist of complexes of the form $\operatorname{Flag} Y_{\mathcal{E}}(V)$ where

- (V, Q, f) is a thick pseudo-quadratic space of Witt index n, in particular $\dim V = 2n$;
- $\mathcal{E} = \mathcal{E}^\perp$ is a finite set of subspaces of V, set $\mathcal{E}_j = \{E \in \mathcal{E} : \dim E = j\}, e_j = |\mathcal{E}_j|$;
- $|K| \geq 2\sum_{j=1}^{2n-1}\binom{2n-2}{j-1}e_j$;
- $\widehat{\mathcal{E}}_n := \mathcal{E}_n \cap \mathcal{U}_n(V)$;
- $X = \{0 < U < V \mid U \text{ is totally isotropic}\}$.

Set

$$Y_{\mathcal{E}}(V) := \{U \in X \mid U \tilde{\pitchfork} \mathcal{E} \text{ and if } \dim U < n \text{ then } U @ \hat{\mathcal{E}}_n\}.$$

Fact 6.3.26. Let $\kappa = \operatorname{Flag} Y_{\mathcal{E}}(V) \in \mathbf{C}_3$ with $\dim V = 2n$. Then there exists a 1-dimensional subspace $L \in Y_{\mathcal{E}}(V)$.

Let $n = \frac{1}{2}\dim V = \dim \kappa + 1$. And assume $n \geq 3$. We construct the filtration for $\kappa = \operatorname{Flag} Y_{\mathcal{E}}(V) \in \mathbf{C}_3$ as follows, using the notation $Y = Y_{\mathcal{E}}(V)$. Set

$$\begin{aligned}
&\mathcal{F} := \mathcal{E} \cup \left(\mathcal{E} \cap L^\perp\right) \cup (\mathcal{E} + L) \cup \left(\left(\mathcal{E} \cap L^\perp\right) + L\right) \text{ and} \\
&\widehat{\mathcal{F}}_n := \mathcal{F}_n \cap \mathcal{U}_n(V).
\end{aligned}$$

Consider the following conditions:

(i) $L \leq U, U \tilde{\pitchfork} \mathcal{E}$ and if $\dim U < n$ then $U @ \widehat{\mathcal{E}}_n$;

(ii) $L \nleq U \leq L^\perp$ and $\begin{cases} U \pitchfork \mathcal{E}, U @ \widehat{\mathcal{E}}_n & \text{if } \dim U = n-1, \\ U \pitchfork \mathcal{E} \cup (\mathcal{E} + L), U @ \widehat{\mathcal{F}}_n & \text{if } \dim U \leq n-2; \end{cases}$

(iii) $U \nleq L^\perp, \dim U > 1, U \tilde{\pitchfork} \mathcal{F}$ and if $\dim U < n$ then $U @ \widehat{\mathcal{F}}_n$.

We define $Y_0 := \{U \in X \mid U \text{ satisfies (i), (ii), or (iii) }\}$ and $\kappa_0 = \operatorname{Flag} Y_0$. The filtration will contain two parts, one dimension increasing and one dimension decreasing filtration. To define the first, we define what we call Condition (iv) in the following way, using the notation from above.

$$U \nleq L^\perp \text{ and } \begin{cases} U \tilde{\pitchfork} \mathcal{F} & \text{if } \dim U = n \\ U \pitchfork \mathcal{F} \text{ and } U @ \widehat{\mathcal{F}}_n & \text{if } \dim U = n-1 \\ U \pitchfork \mathcal{E} \cup (\mathcal{E} + L) \text{ and } U @ \widehat{\mathcal{F}}_n & \text{if } \dim U \leq n-2. \end{cases} \tag{iv}$$

Set

$$\begin{aligned}
Z &:= \{U \in X \mid U \text{ satisfies (i), (ii) or (iv)}\} \\
Y_i &:= \{U \in Z \mid U \in Y_0 \text{ or } \dim U \leq i\}, 1 \leq i \leq n-2.
\end{aligned}$$

Note that $Y_0 \subseteq Z$ since if $\dim U < n$ then $U \tilde{\pitchfork} \mathcal{E} \iff U \pitchfork \mathcal{E}$.

The dimension increasing filtration is defined by setting

$$Y_i := \{U \in Y \mid U \in Y_{n-1} \text{ or } \dim U \geq 2n - i\},\ n \leq i \leq 2n-1$$

and together we get the filtration $\kappa_i = \operatorname{Flag} Y_i$ for $0 \leq i \leq 2n-1$.

As usual, we first show the existence of a $\mathbb{Z}$-cone for κ_0.

Lemma 6.3.27. In the setting above, $\kappa_0 = \operatorname{Flag} Y_0$ has an $(n-2)$-cone function $\operatorname{Cone}_{\kappa_0}$ with

$$\operatorname{Rad}(\operatorname{Cone}_{\kappa_0}) \leq 2n+1.$$

Proof. Let $\kappa = \operatorname{Flag} Y_{\mathcal{E}}(V) \in \mathbf{C}_3$ with $\dim V = 2n$. Let Y_0 and L be defined as above. Recall that $X = \{0 < U < V \mid U \text{ totally isotropic subspace}\}$. We define a filtration for Y_0 and we want to apply Theorem 5.4.6 inductively. We will use the fact stated in Step 3 of [Abr96, Proposition 14], that for every $U \in Y_0$, we have $U \cap L^\perp \in Y_0$ and $(U \cap L^\perp) + L \in Y_0$.

Set

$$\begin{aligned}
&A_0 = \{U \in X \mid U \text{ satisfies (i)}\}\\
&A_i = \{U \in X \mid U \text{ satisfies (i) or } (\dim U \leq i \text{ and } U \text{ satisfies (ii)})\},\ 1 \leq i \leq n;\\
&\alpha_i = \operatorname{Flag} A_i, 0 \leq i \leq n.
\end{aligned}$$

Then $A_0 = \{L\} * \operatorname{Flag}\{U \in A_0 \mid U \neq L\}$ and hence has a cone function with cone radius bounded by 1. Clearly elements in $A_i \setminus A_{i-1}$ have the same dimension and are hence not adjacent. Let $U \in A_i \setminus A_{i-1}$. Then

$$\operatorname{lk}_{\alpha_i}(U) \cap \alpha_{i-1} = \operatorname{Flag}\{W \in X \mid (W \in A_0 \text{ and } U < W) \text{ or } (W \in A_{i-1} \text{ and } W < U)\}.$$

We want to show that each subspace in $\{W \in X \mid (W \in A_0 \text{ and } U < W) \text{ or } (W \in A_{i-1} \text{ and } W < U)\}$ is either contained in or contains $U+L$ (note that we have $U \cap L^\perp + L = U + L \in A_0 \subseteq A_{i-1}$). If $W \in A_0$ and $U < W$ then $L \leq W$ and hence $U + L \leq W$. If $W \in A_{i-1}$ and $W < U$ then $W < U + L$ as needed. The second part of the filtration is defined as follows. We set

$$\begin{aligned}
&B_i = \{U \in X \mid U \in A_n \text{ or } (U \text{ satisfies (iii) and } \dim U \geq n-i+1)\},\\
&\beta_i = \operatorname{Flag} B_i, 0 \leq i \leq n.
\end{aligned}$$

Then $A_n = B_0$. Let $U \in B_i \setminus B_{i-1}$. We want to show that $\operatorname{lk}_{\beta_i}(U) \cap \beta_{i-1}$ can be written as the join of $\{U \cap L^\perp\}$ and $\operatorname{Flag}\{W \in B_{i-1} \mid W \neq U \cap L^\perp, W < U \text{ or } U < W\}$. First, note that $U \cap L^\perp \in Y_0$ and hence $U \cap L^\perp \in A_n \subseteq B_{i-1}$. Let $W \in B_{i-1} \setminus \{U \cap L^\perp\}$. If $W < U$ then $\dim W < n - i$, hence $W \in A_n \setminus A_0$, and in particular $W \leq L^\perp$. Hence $W = W \cap L^\perp \leq U \cap L^\perp$. If $U < W$ then $U \cap L^\perp \leq U \leq W$. Thus we get a filtration of κ_0 of length $2n$. By applying Theorem 5.4.6 at each step, we get an $(n-2)$ cone function $\operatorname{Cone}_{\kappa_0}$ with

$$\operatorname{Rad}(\operatorname{Cone}_{\kappa_0}) \leq 2n+1.$$

□

And we get the analogous result for non-abelian cones.

Lemma 6.3.28. In the setting as above, $\kappa_0 = \operatorname{Flag} Y_0$ has a NAC function Cone_{κ_0} with

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_0}) \leq 2n+1.$$

Proof. We use the same filtration of κ_0 as in the proof of Lemma 6.3.27. Set

$$\begin{aligned}
A_0 &:= \{U \in X \mid U \text{ satisfies (i)}\} \\
A_i &:= \{U \in X \mid U \text{ satisfies (i) or } (\dim U \leq i \text{ and } U \text{ satisfies (ii)})\},\ 1 \leq i \leq n; \\
\alpha_i &:= \operatorname{Flag} A_i, 0 \leq i \leq n.
\end{aligned}$$

Then $A_0 = \{L\} * \operatorname{Flag}\{U \in A_0 \mid U \neq L\}$ and hence has a NAC with cone radius bounded by 1. Thus, by the same argument as above, we can inductively apply Theorem 5.4.3.

The second part of the filtration is defined as follows. We set

$$\begin{aligned}
B_i &:= \{U \in X \mid U \in A_n \text{ or } (U \text{ satisfies (iii) and } \dim U \geq n-i+1)\}, \\
\beta_i &:= \operatorname{Flag} B_i, 0 \leq i \leq n.
\end{aligned}$$

Then $A_n = B_0$. Thus we get a filtration of κ_0 of length $2n$. By applying Corollary 5.4.5 at each step, also for the second part, we get a NAC Cone_{κ_0} with

$$\mathrm{Rad}(\mathrm{Cone}_{\kappa_0}) \leq 2n+1.$$

$\square$

Lemma 6.3.29. Let $\kappa \in \mathbf{C}_3$ of dimension $n \geq 2$ with filtration $\kappa_0 \subset \ldots \kappa_{2n-1} = \kappa$ as above. Set $\mathbf{C}_D = \mathbf{C}_1 \cup \mathbf{C}_2 \cup \mathbf{C}_3$. For every $U \in \kappa_i(0) \setminus \kappa_{i-1}$ we have that

$$\mathrm{lk}_{\kappa_i}(U) \cap \kappa_{i-1} = \kappa_{i-1}^{<U} * \kappa_{i-1}^{>U}$$

is such that $\kappa_{i-1}^{<U}, \kappa_{i-1}^{>U} \in \mathbf{C}_D$ or one (or both) have again a filtration $R_0 \subset \ldots R_h, h \leq \ell_{\dim R_0}$ which starts with an element of $\mathbf{C}_D$ (i.e. $R_0 \in \mathbf{C}_D$), satisfies itself Condition (ii) of Theorem 6.3.10, and is such that for every $W \in R_j(0) \backslash R_{j-1}(0)$ we have $R_{j-1}^{<W}, R_{j-1}^{>W} \in \mathbf{C}_D$.

Proof. We start with the first part of the filtration. Recall

$$\begin{aligned}
Z &:= \{U \in X \mid U \text{ satisfies (i), (ii) or iv }\} \\
Y_i &:= \{U \in Z \mid U \in Y_0 \text{ or } \dim U \leq i\}, 1 \leq i \leq n-2.
\end{aligned}$$

Note that $Y_0 \subseteq Z$ since if $\dim U < n$ then $U \tilde{\pitchfork} \mathcal{E} \iff U \pitchfork \mathcal{E}$.

Let $U \in Y_i \setminus Y_{i-1}$. We have that $U \not\leq L^\perp$ and $\dim U = i \leq n-2$. We get $Y_{i-1}^{<U} = Z^{<U}$. Since $L \not\leq U$ (otherwise $U \leq U^\perp \leq L^\perp$), we can apply Step 5 of the proof of [Abr96, Proposition 14] to get that $\operatorname{Flag} Z^{<U} \in \mathbf{C}_2$.

On the other hand, $Y_{i-1}^{>U} = Y_0^{>U}$ and we have $\operatorname{Flag} Y_0^{>U} \in \mathbf{C}_3$ by [Abr96].

Next, we treat $U \in Y$ with $\dim U = n-1$. To cover these elements, we define

$$Y_{n-1} := Z \cup \{U \in Y \mid \dim U = n-1 \text{ and } Y_0^{>U} \neq \emptyset\}.$$

Let $U \in Y_{n-1} \setminus Y_{n-2}$. Then $Y_{n-2}^{<U} = Z^{<U}$ and $\operatorname{Flag} Z^{<U} \in \mathbf{C}_2$. Furthermore, we have $Y_{n-2}^{>U} = Y_0^{>U} \in \mathbf{C}_3$. (Note that $\dim \operatorname{Flag} Y_0^{>U} = 0$ since $Y_0^{>U}$ only contains subspaces of dimension n. Since $Y_0^{>u} \neq \emptyset$ it trivially admits a (-1)-cone function.)

Recall that the second part of the filtration is given by

$$Y_i := \{U \in Y \mid U \in Y_{n-1} \text{ or } \dim U \geq 2n - i\}, \kappa_i = \operatorname{Flag} Y_i, n \leq i \leq 2n-1.$$

Let $U \in Y_i \setminus Y_{i-1}$ and set $k = 2n - i = \dim U$. We distinguish the cases $k = n$ and $k < n$.

If $k = n$, then $Y_{n-1}^{>U} = \emptyset$ since there is no totally isotropic subspace of dimension $> n$. This is fine for our purpose, since $\emptyset * A = A$ for an arbitrary simplicial complex A. On the other hand, $\operatorname{Flag} Y_{n-1}^{<U}$ has again a filtration starting from an element in $\mathbf{C}_1$. To describe the filtration, we set $\mathcal{E}' := (\mathcal{E} \cap U) \cup ((\mathcal{E} + L) \cap U), \mathcal{F}' := \mathcal{F} \cap U, \mathcal{H} := \mathcal{H}_1 := \cdots := \mathcal{H}_{n-2} = \widehat{\mathcal{F}_n}$ and $\mathcal{H}_{n-1} := \{F \in \widehat{\mathcal{F}}_n \mid U \cap F = 0\}$. Furthermore, we set

$$S := X_{\mathcal{E}';\mathcal{H}}(U;V) \cap \{0 < W < U \mid F' \not\leq W \text{ for all } 0 \neq F' \in \mathcal{F}'\}.$$

By Step (20) in [Abr96, Proposition 14], we know that $S \subseteq Y_{n-1}^{<U}$ For $\operatorname{Flag} S$, we get the desired filtration as follows. Note that if $W \pitchfork_U F'$ then this implies that $F' \not\leq W$. Thus we set $\mathcal{E}'' = \mathcal{E}' \cup \mathcal{F}'$ and get $X_{\mathcal{E}'',\mathcal{H}}(U;V) \subseteq S$. We set

$$A_0 := X_{\mathcal{E}'',\mathcal{H}}(U;V), A_i = \{W \in S \mid W \in A_0 \text{ or } \dim W \leq i\}, 1 \leq i \leq n-1.$$

We follow the by now standard procedure to check that this is a valid filtration. Let $W \in A_i \setminus A_{i-1}$. Then $A_{i-1}^{>W} = A_0^{>W} = X_{\mathcal{E}'',\mathcal{H}}(U;V)^{>W}$ and by Step 4. in [Abr96, Lemma 33] $\operatorname{Flag} X_{\mathcal{E}'',\mathcal{H}}(U;V)^{>W} \in \mathbf{C}_1$. On the other hand, $A_{i-1}^{<W} = S^{<W}$. Note that if $W \in S$ and $W' \in X_{\mathcal{E}';\mathcal{H}}(U;V)$ such that $W' \leq W$ then $W' \in S$ since if $F' \not\leq W$ then $F' \not\leq W' \leq W$. Thus

$$\begin{aligned} S^{<W} &= X_{\mathcal{E}';\mathcal{H}}(U;V)^{<W} = \{0 < W' < W \mid W' \cap \mathcal{E}', W' @ \mathcal{H}_j \text{ for } \dim W' = j\} \\ &= X_{\mathcal{E}' \cap W,\mathcal{H}}(W;V) \end{aligned}$$

and $\operatorname{Flag} X_{\mathcal{E}' \cap W,\mathcal{H}}(W;V) \in \mathbf{C}_1$. Hence we got a filtration

$$\operatorname{Flag} A_0 \subseteq \ldots \subseteq \operatorname{Flag} A_{n-1} = \operatorname{Flag} S \subseteq \operatorname{Flag} Y_{n-1}^{<U}$$

of length $n = \dim U$ which satisfies the assumptions.

If $\dim U < n$, then $Y_{i-1}^{<U} = Z^{<U}$ and we again have $\operatorname{Flag} Z^{<U} \in \mathbf{C}_3$. Additionally, we have $Y_{i-1}^{>U} = Y^{>U}$ for which Abramenko shows that $\operatorname{Flag} Y^{>U} \in \mathbf{C}_3$. □

We now have all the necessary ingredients to show that the relevant class of simplicial complexes satisfies the conditions of Theorems 6.3.10 and 6.3.11. The proof will follow the usual structure.

Theorem 6.3.30. The class $\mathbf{C}_D = \mathbf{C}_1 \cup \mathbf{C}_2 \cup \mathbf{C}_3$ satisfies the conditions of Theorems 6.3.10 and 6.3.11.

Proof. Let $\kappa = \operatorname{Flag} Y \in \mathbf{C}_D$. If $\kappa \in \mathbf{C}_1 \cup \mathbf{C}_2$ then the result follows from Corollaries 6.3.18 and 6.3.24.

Now assume $\kappa = \operatorname{Flag} Y_{\mathcal{E}}(V) \in \mathbf{C}_3$. Let $n = \frac{1}{2} \dim V = \dim \kappa + 1$. We will show that κ satisfies the required conditions for all $n \geq 2$.

Let $n = 2$, hence $\dim \kappa = 1$. Step 2. in [Abr96, Proposition 14] shows that each vertex $U \in Y$ can be connected to L by a path of length at most 5. Thus these paths define a non-abelian 0-cone function (and hence also $(\mathbb{Z}, 0)$-cone function) Cone_κ of κ with radius at most 5.

For $n \geq 3$, this follows using the filtration discussed above, together with Lemmas 6.3.27 to 6.3.29. $\square$

The following remark from [Abr96] gives the connection between $X_{\mathcal{E}}(V)$ and $Y_{\mathcal{E}}(V)$.

Remark 6.3.31. Let $\tau = \{E_1, \dots E_r\}$ be a simplex of $\tilde{\Delta} = \operatorname{Orifl} \tilde{X}$ and set $\mathcal{E} = \mathcal{E}(a) = \{E_i, E_i^{\perp} \mid 1 \leq i \leq r\}$. Then $\mathcal{E} \cap \mathcal{U}_n = \emptyset$, $Y_{\mathcal{E}}(V) = X_{\mathcal{E}}(V)$, $e_n \leq 2, e_{n-1} = e_{n+1} = 0$ and $e_i \leq 1$ for all other $1 \leq i \leq 2n-1$.

We conclude with two corollaries, capturing the essential result of the section, once for the abelian and once for the non-abelian case.

Corollary 6.3.32. Let $\tilde{\Delta}$ be a building of type D_n over the field K, and $a \in \tilde{\Delta}$ be a simplex. If $|K| \geq 2^{2n-1}$, then $\tilde{\Delta}_{\mathrm{op}}(a)$ has a $(\mathbb{Z}, n-2)$-cone function with $\mathrm{Rad}(\mathrm{Cone}_{\tilde{\Delta}_{\mathrm{op}}(a)}) \leq 2\mathcal{R}(n-1)$, where $\mathcal{R}(n)$ does not depend on K.

Proof. By Theorem 6.3.30, we have that $Y_{\mathcal{E}(a)}(V)$ has a cone function with radius $\leq \mathcal{R}(n-1)$. By Remark 6.3.31, we have $X_{\mathcal{E}(a)}(V) = Y_{\mathcal{E}(a)}(V)$. Thus Proposition 6.3.1 yields a cone function for $\tilde{\Delta}_{\mathrm{op}}(a) = \tilde{T}_{\mathcal{E}(a)}(V)$ with cone radius bounded by $2\mathcal{R}(n-1)$. $\square$

Corollary 6.3.33. Let $\tilde{\Delta}$ be a building of type D_n over the field K, and $a \in \tilde{\Delta}$ be a simplex. If $|K| \geq 2^{2n-1}$, then $\tilde{\Delta}_{\mathrm{op}}(a)$ has a NAC with $\mathrm{Rad}_j(\mathrm{Cone}_{\tilde{\Delta}_{\mathrm{op}}(a)}) \leq 2\mathcal{R}(n-1), 0 \leq j \leq 1$, where $\mathcal{R}(n)$ does not depend on K.

Proof. By Theorem 6.3.30, we have that $Y_{\mathcal{E}(a)}(V)$ has NAC with radius $\leq \mathcal{R}(n-1)$. By Remark 6.3.31, we have $X_{\mathcal{E}(a)}(V) = Y_{\mathcal{E}(a)}(V)$. Thus Proposition 6.3.2 yields a NAC for $\tilde{\Delta}_{\mathrm{op}}(a) = \tilde{T}_{\mathcal{E}(a)}(V)$ with cone radius bounded by $2\mathcal{R}(n-1)$. $\square$

6.4 Vanishing of first cohomology

In this chapter, we show that the first cohomology of the congruence KMS complexes vanishes for many choices of coefficient groups. Together with the fact that these complexes have bounded h^1_{cs}, this will show that the congruence KMS complexes give rise to families of bounded degree 1-coboundary expanders.

The idea is to apply [KOW24, Theorem 3.1] which reduces the question of vanishing of 1-cohomology for each KMS complex to the vanishing of 1-cohomology for an underlying infinite complex and the vanishing of the 1-cohomology of the kernels of the quotient maps. Before we recall the precise statement, we give two technical definitions.

Definition 6.4.1 ([Ser97, Chapter I.5])**.** Let G, Λ be (potentially non-abelian) groups, and assume that G acts on Λ by automorphisms. Then we define the 0-*cohomology of* G *with coefficient group* Λ as

$$H^0(G, \Lambda) = \{x \in \Lambda \mid g.x = x \text{ for all } g \in G\}.$$

We further define the set of 1-*cocycles* as

$$Z^1(G, \Lambda) = \{f : G \to \Lambda \mid f(gh) = f(g) \cdot g.f(h)\}$$

and we say that two cocycles f, f' are cohomologous, denoted $f \sim_{\text{coho}} f'$, if there exists $x \in \Lambda$ such that $f'(g) = x^{-1}f(g)(g.x)$. The first cohomology is then the set of equivalence classes of cocycles, i.e. $H^1(G, \Lambda) = Z^1(G, \Lambda)/{\sim_{\text{coho}}}$.

Note that if the action of G on Λ is trivial (i.e. $g.x = x$ for all $g \in G, x \in \Lambda$), then $Z^1(G, \Lambda)$ is precisely the set of group homomorphisms from G to Λ.

Definition 6.4.2. Let X be a simplicial complex and let $N \leq \text{Aut}(X)$ be a group acting on X. The quotient $N\backslash X$ is the simplicial complex defined as follows. We define an equivalence relation $\sim$ on the vertices $V(X)$ of X by setting $v \sim u$ if and only if there exists $g \in N$ such that $g.v = u$. The set of vertices of $N\backslash X$ is the set of equivalence classes, i.e.

$$V(N\backslash X) = V(X)/_{\sim} = \{[v] \mid v \in V(X)\}.$$

A set of vertices $\{[v_0], \ldots, [v_k]\}$ forms a k-simplex in $N\backslash X$ if there exist $u_i \in [v_i]$ such that $\{u_0, \ldots, u_k\} \in X(k)$. The *projection map* $p : X \to N\backslash X$ is the map induced by $p(v) = [v]$ for all $v \in V(X)$. It is always simplicial and surjective. We say that it is *rigid* if for all $\sigma \in X$ we have $\dim(\sigma) = \dim(p(\sigma))$.

Theorem 6.4.3 ([KOW24])**.** Let X be a connected pure n-dimensional simplicial complex, $n \geq 2$ and N be a group acting on X. Assume $p : X \to N\backslash X$ is rigid. For every group Λ, if $H^1(X, \Lambda) = 0$ and $H^1(N, \Lambda) = 0$ then $H^1(N\backslash X, \Lambda) = 0$ (the cohomology of N is taken with the trivial action of N on Λ).

In the remainder of the section, we will first investigate the underlying infinite complex in our setting, and then the kernel of the quotient map.

The underlying infinite complex The complex that will play the role of X in Theorem 6.4.3 in our setting is the following.

Definition 6.4.4. Given an n-spherical, purely n-spherical GCM $A = (A_{ij})_{i,j \in I}$ of rank $n+1$ and a field K, we define the *fundamental KMS complex* to be

$$Y_A(K) = \mathcal{CC}(\mathcal{U}_A(K), (U_{I\setminus\{i\}})_{i \in I}).$$

Observation 6.4.5. The complex $Y_A(K)$ is a connected, pure n-dimensional simplicial complex. If $n \geq 3$ and $|K| \geq 5$ (i.e. $U^+ \cong \mathcal{U}_A(K)$), then it is isomorphic to the opposite complex of the twin building associated to $G_A(K)$.

Proposition 6.4.6. Let A be an n-spherical, purely n-spherical GCM of rank $n+1$, $n \geq 2$, then $Y_A(K)$ is simply connected, in particular $H^1(Y_A(K), \Lambda) = 0$ for every group Λ.

Proof. This follows from Proposition 2.4.7 since $\mathcal{U}_A(K)$ can be described as the free product of the $U_{I\setminus\{i\}}$ amalgamated along their intersections. $\square$

Lemma 6.4.7. Let $\phi_f : \mathcal{U}_A(K) \to \mathrm{Chev}_{\mathring{A}}(K[t]/(f))$ be as in Section 4.3. Let $N_f = \ker \phi_f$. Then $N_f \backslash Y_A(K) \cong X(\mathring{\Phi}, K, f)$ and $p_f : Y_A(k) \to N_f \backslash Y_A(K)$ is rigid.

Proof. This follows from [KOW24, Proposition 2.5] and [KOW24, Observation 2.2]. $\square$

The kernel of the quotient map Let A by a generalized Cartan matrix of untwisted affine type and rank $n \geq 4$, and let K be a finite field with $|K| \geq 5$. For simplicity, we assume that $\mathrm{char}(K) \neq 2$, see Remark 4.3.7. Let $\mathring{A}$ denote the spherical GCM related to A as Section 4.3, with root system $\mathring{\Phi}$. Recall from Section 4.3 that we have an injective homomorphism

$$\begin{aligned} \phi : \mathcal{U}_A(K) &\to \mathrm{Chev}_{\mathring{A}}(K[t]) \\ u_{\alpha_i}(\lambda) &\mapsto \begin{cases} x_{\alpha_i}(\lambda) & i \neq 0 \\ x_{-\gamma}(\lambda t) & i = 0 \end{cases} \end{aligned}$$

where γ is the highest root in $\mathring{\Phi}$.

Let $\mathcal{I}$ denote the image of ϕ in $\mathrm{Chev}(K[t])$ (since the type of the Chevalley group is fixed here, we will omit it from the notation). As shown in Proposition 4.3.8, we have

$$\mathcal{I} = \langle x_\alpha(g) \mid \alpha \in \mathring{\Phi}^+ \text{ and } g \in K[t], \text{ or } \alpha \in \mathring{\Phi}^- \text{ and } g \in (t) \rangle \leq \mathrm{Chev}(K[t]).$$

Furthermore, let $f \in K[t]$ be irreducible with $\deg(f) \geq 2$.

The quotient $K[t] \to K[t]/(f)$ induces the map $\pi_f : \mathrm{Chev}(K[t]) \to \mathrm{Chev}(K[t]/(f))$. We will denote its kernel by $\mathrm{Chev}(K[t], (f)) := \ker(\pi_f)$.

We introduce some further notation, where for a group G and a subset S we denote by $\langle\langle S \rangle\rangle_G$ the normal subgroup of G generated by S. This will involve the Steinberg group of type $\mathring{A}$ over $K[t]$, $\mathrm{St}(K[t]) = \mathrm{St}_{\mathring{A}}(K[t])$.

$$\begin{aligned} \mathcal{I}(f) &= \langle\langle x_\alpha(g) \mid \alpha \in \mathring{\Phi}^+ \text{ and } g \in (f), \text{ or } \alpha \in \mathring{\Phi}^- \text{ and } g \in (tf) \rangle\rangle_{\mathcal{I}}, \\ \mathrm{St}\,(\mathcal{I}) &= \langle x_\alpha(g) \mid \alpha \in \mathring{\Phi}^+ \text{ and } g \in K[t], \text{ or } \alpha \in \mathring{\Phi}^- \text{ and } g \in (t) \rangle \leq \mathrm{St}\,(K[t]), \\ \mathrm{St}\,(\mathcal{I}, (f)) &= \langle\langle x_\alpha(g) \mid \alpha \in \mathring{\Phi}^+ \text{ and } g \in (f), \text{ or } \alpha \in \mathring{\Phi}^- \text{ and } g \in (tf) \rangle\rangle_{\mathrm{St}(\mathcal{I})}. \end{aligned}$$

The following result is of a similar flavor as [ACR12, Lemma 3.3] and the proof is inspired by the proof of the aforementioned result.

Theorem 6.4.8. In the above set up, we have

$$\mathrm{Chev}(K[t], (f)) \cap \mathcal{I} = \mathcal{I}(f).$$

Proof. We have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r, "\alpha"] \arrow[d] & 0 \arrow[d] \\
\mathrm{St}(\mathcal{I},(f)) \arrow[r, "\beta"] \arrow[d] & \mathrm{Chev}(K[t],(f)) \cap \mathcal{I} \arrow[d] \\
\mathrm{St}(\mathcal{I}) \arrow[r, "\gamma"] \arrow[d] & \mathcal{I} \arrow[d] \\
\mathrm{St}(K[t]/(f)) \arrow[r, "\delta"] & \mathrm{Chev}(K[t]/(f))
\end{tikzcd}$$

Here, α is the identity, β, γ and δ are given by mapping a generator $x_\alpha(g)$ of the Steinberg group to the corresponding generator (with the same label) in the Chevalley group.

Note that $\mathcal{I}(f) = \beta(\mathrm{St}(\mathcal{I},(f))) \subseteq \mathrm{Chev}(K[t],(f)) \cap \mathcal{I}$.

The vertical maps in the diagram are first the trivial map, then the inclusion map followed by the quotient maps $\mathrm{St}(K[t]) \to \mathrm{St}(K[t]/(f))$ and $\mathrm{Chev}(K[t]) \to \mathrm{Chev}(K[t]/(f))$ respectively, both restricted to the desired domain.

Our goal is to show that β is surjective. To this end, we want to apply the "Four lemma" (see e.g. [Mac63, Lemma I.3.2]) which says that if the columns are exact, δ is injective and α and γ are surjective then β is surjective.

Note that α is trivially surjective, δ is an isomorphism, since $K[t]/(f)$ is a finite field (see Remark 3.2.5) and γ is surjective since all generators of $\mathcal{I}$ lie in the image of $\mathrm{St}(\mathcal{I})$.

It remains to show the exactness of the columns. For the right column, this follows from the definitions.

For the left column we need to show that $M := \ker(\mathrm{St}\,(\mathcal{I}) \to \mathrm{St}\,(K[t]/(f)))$ is equal to $N := \mathrm{St}\,(\mathcal{I},(f))$.

Clearly we have that $N \subseteq M$. For the other inclusion, we proceed in two steps.
Step 1: We show that $\mathrm{St}\,(\mathcal{I})\,/N \cong \mathrm{St}\,(\mathcal{I})\,/M$.

Let $\psi : \mathrm{St}\,(\mathcal{I}) \to \mathrm{St}\,(\mathcal{I})\,/N$ denote the projection. For $\alpha \in \mathring{\Phi}^+, g \in K[t], h \in (f)$ we have

$$\psi(x_\alpha(g+h)) = \psi(x_a(g))\psi(x_\alpha(h)) = \psi(x_\alpha(g))$$

since $x_\alpha(h) \in N$. Similarly, for $\alpha \in \mathring{\Phi}^-, g \in (t), h \in (tf)$ we have

$$\psi(x_\alpha(g+h)) = \psi(x_a(g))\psi(x_\alpha(h)) = \psi(x_\alpha(g)).$$

Thus the expressions $\psi(x_\alpha(g+(f)))$ for $\alpha \in \mathring{\Phi}^+, g \in K[t]$ and $\psi(x_\alpha(g+(tf)))$ for $\alpha \in \mathring{\Phi}^-, g \in (t)$ are well defined. In particular, they form a set of generators of $\mathrm{St}\,(\mathcal{I})\,/N$.

Let $\mathrm{ht}(\alpha)$ denote the height of the root α with respect to a fixed set of simple roots (i.e. if $\mathring{\Pi} = \{\alpha_1, \ldots, \alpha_n\}$ and $\alpha = \sum_{i=1}^n \lambda_i \alpha_i$ then $\mathrm{ht}(\alpha) = \sum_{i=1}^n \lambda_i$). Note that if $\alpha \in \mathring{\Phi}^+$ then $\mathrm{ht}(\alpha) > 0$ and if $\alpha \in \mathring{\Phi}^-$ then $\mathrm{ht}(\alpha) < 0$ (and the height is always an integer).

We define the following map:

$$\xi : \mathrm{St}\,(K[t]/(f)) \to \mathrm{St}\,(\mathcal{I})\,/N$$
$$x_\alpha(g+(f)) \mapsto \begin{cases} \psi(x_\alpha(t^{-\operatorname{ht}(\alpha)}g+(f))) & \text{if } \alpha \in \mathring{\Phi}^+ \\ \psi(x_\alpha(t^{-\operatorname{ht}(\alpha)}g+(tf))) & \text{if } \alpha \in \mathring{\Phi}^- \end{cases}.$$

Note that if $\alpha \in \mathring{\Phi}^-$ then $-\operatorname{ht}(\alpha) > 0$ and thus $t^{-\operatorname{ht}(\alpha)} \in K[t]$. In particular, if $g \in (f)$ then $t^{-\operatorname{ht}(\alpha)}g \in (tf)$. If $\alpha \in \mathring{\Phi}^+$ then $-\operatorname{ht}(\alpha) < 0$. We choose $\tilde{g}_\alpha \in K[t]$ such that $\tilde{g}_\alpha t^{\operatorname{ht}(\alpha)} = 1 \mod (f)$ i.e. $\tilde{g}_\alpha + (f) = t^{-\operatorname{ht}(\alpha)} + (f)$. Hence by $t^{-\operatorname{ht}(\alpha)}g + (f)$ we mean $\tilde{g}_\alpha g + (f)$. In particular, if $g \in (f)$ then $\tilde{g}_\alpha g \in (f)$. Therefore, different representatives of the same class $g + (f)$ get mapped to the same element by ξ.

Thus, to check that ξ is a well-defined homomorphism, it remains to check that the relations are respected by the map.

Clearly, $\xi(x_\alpha(s+u)) = \xi(x_\alpha(s))\xi(x_\alpha(u))$. Note that $\operatorname{ht}(i\alpha+j\beta) = i\operatorname{ht}(\alpha) + j\operatorname{ht}(b)$ for all $\alpha, \beta \in \mathring{\Phi}, i, j \in \mathbb{Z}$ and hence

$$[x_\alpha(t^{-\operatorname{ht}(\alpha)}s), x_\beta(t^{-\operatorname{ht}(\beta)}u)] = \prod_{i\alpha+j\beta\in\mathring{\Phi}, i,j>0} x_{i\alpha+j\beta}(C^{\alpha,\beta}_{i,j} t^{-\operatorname{ht}(i\alpha+j\beta)} s^i u^j).$$

This implies that the commutator relation is also preserved under ξ.

Thus, ξ is a well-defined homomorphism. The next step is to show that it is surjective. This follows from the fact that

$$K[t]/(f) \to K[t]/(f) : \lambda \mapsto (t + (f))^n \lambda$$

is bijective for all $n \in \mathbb{Z}$ and that the map

$$K[t]/(f) \to (t)/(tf) : \lambda \mapsto t\lambda$$

is well defined and bijective. Hence all the generators of $\mathrm{St}\,(\mathcal{I})\,/N$ lie in the image of ξ.

On the other hand, since $N \subseteq M$, the surjective map $\mathrm{St}\,(\mathcal{I}) \to \mathrm{St}\,(K[t]/(f))$ induces a surjective map $\mathrm{St}\,(\mathcal{I})\,/N \to \mathrm{St}\,(K[t]/(f))$. Since $\mathrm{St}\,(K[t]/(f))$ is finite and $\mathrm{St}\,(K[t]/(f)) \cong \mathrm{St}\,(\mathcal{I})\,/M$ we get that $\mathrm{St}\,(\mathcal{I})\,/N \cong \mathrm{St}\,(\mathcal{I})\,/M$, which concludes Step 1.

Step 2: Since $\mathrm{St}\,(\mathcal{I})\,/M \cong \mathrm{St}\,(K[t]/(f)) \cong \mathrm{Chev}(K[t]/(f))$ is finite, we can conclude from Step 1 that $N = M$.

To show this, note that we have $N, M \lhd \mathrm{St}\,(\mathcal{I})\,, N \subseteq M$ thus $M/N \lhd \mathrm{St}\,(\mathcal{I})\,/N$. Hence by the third isomorphism theorem $(\mathrm{St}\,(\mathcal{I})\,/N)/(M/N) \cong \mathrm{St}\,(\mathcal{I})\,/M$. Hence $|\mathrm{St}\,(\mathcal{I})\,/N|/|M/N| = |\mathrm{St}\,(\mathcal{I})\,/M|$. But $|\mathrm{St}\,(\mathcal{I})\,/N| = |\mathrm{St}\,(\mathcal{I})\,/M|$, thus $|M/N| = 1$ which implies $N = M$.

□

Corollary 6.4.9. Assume the setting from above. Let $p = \operatorname{char}(K)$ and let Λ be a group with no non-trivial elements of order p. Then $H^1(\ker(\phi_f), \Lambda) = 0$, where the cohomology is considered with trivial action of $\ker(\phi_f)$ on Λ.

Proof. By Theorem 6.4.8, we have that $\mathrm{Chev}(K[t],(f))\cap\mathcal{I}$ is normally generated by order p elements, and hence the same holds for $B := \ker(\phi_f)$ (since the two groups are isomorphic because ϕ is injective). Since Λ has no non-trivial elements of order p the only possible morphism in $\hom_{\mathrm{Grp}}(B,\Lambda)$ is the trivial one. This implies that $H^1(B,\Lambda)$ is trivial, since the elements of the first non-abelian cohomology are precisely conjugacy classes of the elements of $\hom_{\mathrm{Grp}}(B,\Lambda)$. Note that if Λ is abelian, then the two sets coincide. □

The vanishing of first cohomology of congruence KMS complexes is summarized in the following corollary.

Corollary 6.4.10. Let $\mathring{\Phi}$ be a spherical root system of rank $n \geq 3$, K a finite field with $|K| \geq 5$ and $\mathrm{char}(K) = p \neq 2$. Let $f \in K[t]$ be an irreducible polynomial of degree at least 2. Let $X(\mathring{\Phi},K,f)$ denote the corresponding congruence KMS complex as in Definition 4.3.10. Furthermore, let Λ be a group with no non-trivial elements of order p. Then

$$H^1(X(\mathring{\Phi},K,f),\Lambda) = 0.$$

Proof. This follows from Theorem 6.4.3 together with Proposition 6.4.6, Lemma 6.4.7 and Corollary 6.4.9. □

Remark 6.4.11. A similar argument, using Theorem 6.4.3 and Proposition 6.4.6, could be applied to other finite quotients of KMS groups used in the KMS complex construction as well, provided that one can show that the cohomology of the kernel of the quotient is trivial.

6.5 Coboundary and cosystolic expansion results for KMS complexes

We are now able to harvest to fruits of our work and conclude that certain KMS complexes give rise to cosystolic and coboundary expanders.

Theorem 6.5.1. Let $n \geq 3$ be an integer. There exists $\varepsilon > 0$ such that for every prime power $q > 2^{2n-1}$ and every n-dimensional, n-classical KMS complex X over $\mathbb{F}_q$, all proper links of X are $(\varepsilon,\mathbb{A})$-coboundary expanders for every finitely generated abelian group $\mathbb{A}$.

Proof. Fix a rank $n+1$, n-classical GCM $A = (A_{ij})_{i,j\in I}$. Let $\sigma \in X$ be of type $\emptyset \neq T \subsetneq I$ and dimension $\leq n-2$. Then $\mathrm{lk}_X(\sigma)$ is isomorphic to the opposite complex in the spherical building of type $A_{I\setminus T}$ over $\mathbb{F}_q$ by Corollary 4.2.9. Since A is n-classical this is the join of opposite complexes of type A_k, C_k or D_k. Thus, Corollaries 6.1.5, 6.2.6 and 6.3.32 together with Proposition 5.3.12 and Theorem 5.2.7 imply that $\mathrm{lk}_X(\sigma)$ is a $(\varepsilon_\sigma,\mathbb{Z})$-coboundary expander for some ε_σ. Since, after fixing n, there are only finitely many cases to check (still fixing the group $\mathbb{Z}$), we let ε be the minimum over all ε_σ. By Proposition 5.2.14, this also implies $(\varepsilon,\mathbb{A})$-coboundary expansion for every finitely generated abelian group $\mathbb{A}$. □

We also get the analogous result for non-abelian expansion.

Theorem 6.5.2. Let $n \geq 3$ be an integer. There exists $\varepsilon > 0$ such that for every prime power $q > 2^{2n-1}$ and every n-dimension, n-classical KMS complex X over $\mathbb{F}_q$, all proper links of X are (Λ, β) 1-coboundary expanders for every group Λ.

Proof. Fix a rank $n+1$, n-classical GCM $A = (A_{ij})_{i,j \in I}$. Let $\sigma \in X$ be of type $\emptyset \neq T \subsetneq I$ and dimension $\leq n-2$. Then $\mathrm{lk}_X(\sigma)$ is isomorphic to the opposite complex in the spherical building of type $A_{I \setminus T}$ over $\mathbb{F}_q$ by Corollary 4.2.9. Since A is n-classical this is the join of opposite complexes of types A_k, C_k or D_k. Thus, Corollaries 6.1.6, 6.2.6 and 6.3.33 together with Section 5.3 and Proposition 5.1.8 imply that $\mathrm{lk}_X(\sigma)$ is a (λ, β_σ) 1-coboundary expander for some β_σ. Since, after fixing n, there are only finitely many cases to check (note that the expansion constant does not depend on the coefficient group since it comes from the cone radius), we let β be the minimum over all β_σ. □

Remark 6.5.3. If we consider a KMS complex over the GCM of type $\tilde{A}_n$, then we get a smaller bound on the field size, namely $q > 2^{n-1}$.

Using the local-to-global results for cosystolic expansion, Theorem 2.3.16, we can conclude the following.

Corollary 6.5.4. Let $n \geq 3$ be an integer and q a prime power. Let A be a rank $n+1$, n-classical GCM and let X be a KMS complex of type A over $\mathbb{F}_q$. Let Y denote the $(n-1)$-skeleton of X. Then there exist $\varepsilon, \mu > 0, q_0 \in \mathbb{N}$ such that if $q \geq q_0$ then Y is an $(\varepsilon, \mu, \mathbb{A})$-cosystolic expander for every finitely generated abelian group $\mathbb{A}$.

Proof. This follows from Theorem 2.3.16, since by Theorem 4.4.3 X is a local spectral expander, where the expansion constant can be made arbitrary good by increasing the size of the field. On the other hand, by Theorem 6.5.1, all proper links are coboundary expanders, and the expansion constant is independent of q and hence the spectral expansion constant. □

Corollary 6.5.5. Let $n \geq 3$ be an integer and q a prime power. Let A be a rank $n+1$, n-classical GCM and let $\{X_i\}_i$ be an infinite family of KMS complex of type A over $\mathbb{F}_q$. Let Y_i denote the $(n-1)$-skeleton of X_i. Then there exist $\varepsilon, \mu > 0, q_0 \in \mathbb{N}$ such that if $q \geq q_0$ then $\{Y_i\}_i$ is an infinite family of bounded degree $(\varepsilon, \mu, \mathbb{A})$-cosystolic expander for every finitely generated abelian group $\mathbb{A}$.

In particular, $\{Y_i\}_i$ is a family of bounded degree topological expanders.

In the non-abelian case, we obtain the following.

Corollary 6.5.6. Let $n \geq 3$ be an integer. There exist $\varepsilon > 0, q_0 \in \mathbb{N}$ such that for all prime powers $q > q_0$ and every n-dimensional, n-classical KMS complex X over $\mathbb{F}_q$ and every group Λ, we have $h^1_{cs}(X, \Lambda) \geq \varepsilon$.

Proof. This follows from Theorem 2.3.22, since by Theorem 4.4.3 X is a local spectral expander and by Theorem 6.5.2 the relevant links are coboundary expanders. □

Combining this with the results from Section 6.4 we obtain 1-coboundary expansion for the congruence KMS complexes.

Corollary 6.5.7. Let $n \geq 3$ and let $\mathring{\Phi}$ be a root system of type A_n, B_n, C_n or D_n. Then there exist $q_0 \in \mathbb{N}, \varepsilon > 0$ such that

(i) for every field K with $|K| \geq q_0$ and $\mathrm{char}(K) = p \neq 2$,

(ii) for every group Λ with no non-trivial elements of order p, (e.g. if Λ is finite and $|\Lambda| < p$),

(iii) for every irreducible polynomial $f \in K[t]$ of degree at least 2

the congruence KMS complex $X(\mathring{\Phi}, K, f)$ is a (Λ, ε) 1-coboundary expander.

Proof. Let $X_f = X(\mathring{\Phi}, K, f)$. Then by Corollary 6.5.6, we get $q_0 \in \mathbb{N}, \varepsilon > 0$ such that if $q \geq q_0$ then $h^1_{cs}(X_f, \Lambda) \geq \varepsilon$. From Corollary 6.4.10 we obtain $h^1_{cs}(X_f, \Lambda) = h^1_{cb}(X_f, \Lambda)$ given that Λ has no non-trivial elements of order p. □

Corollary 6.5.8. Let $n \geq 3$ and let $\mathring{\Phi}$ be a root system of type A_n, B_n, C_n or D_n. Let $q_0 \in \mathbb{N}, \varepsilon > 0$ be the constants from Corollary 6.5.7. Let K be a field with $|K| \geq q_0$ and $\mathrm{char}(K) = p \neq 2$, and let $f_i \in K[t]$ be a family of irreducible polynomial such that $\lim_{i \to \infty} \deg(f_i) = \infty$. Let Λ be a group with no non-trivial elements of order p. Then $\{X(\mathring{\Phi}, K, f_i)\}_i$ is an infinite family of bounded degree (ε, Λ) 1-coboundary expanders.

Chapter 7

Directions for future research

In this last chapter, we collected some questions and ideas for future research around high-dimensional expansion and KMS complexes.

7.1 Higher dimensional coboundary expanders

Up to this point, the only known families of bounded degree coboundary expanders are two dimensional, including the ones we get from the congruence KMS complexes. Or differently said, we only know 1-coboundary expansion for the classical congruence KMS complexes. Naturally, the question of 2- (and higher) coboundary expansion arises.

One possible strategy to answer this question could be as follows. We know from the work of Abramenko in [Abr96], that many opposite complexes are not just simply connected, but $(n-2)$-connected (where n is the dimension). Together with a high-dimensional version of Theorem 6.4.3 it remains to show that the second group cohomology of the kernel of the quotient maps vanishes.

Theorem 7.1.1. Let A be a rank $n+1$, n-classical GCM, K a finite field with $|K|\geq 2^{2n-1}$ (or 2^{n-1} if A is of type $\tilde{A}_{n+1}$). Let $(\Delta^+, \Delta^-, \delta^*)$ be the twin building associated to $G = G_A(K)$. Let $C^+ = 1B^+$ be the positive fundamental chamber. Then Δ^-_{op} is $(n-2)$-connected.

Proof. Note that U^+ stabilizes C^+. Let $X = |\Delta^-|$ be the geometric realization of the negative building. For $\sigma, \tau \in \Delta^-$ let

$$d(\sigma, \tau) = \min\left\{\mathrm{length}(\delta^-(C, D)) \mid C, D \in \mathcal{C}^-, \tau \subseteq C, \sigma \subseteq D\right\}$$

and for $A, B \subseteq \Delta^-$ let $d(A, B) = \min_{a\in A, b\in B} d(a, b)$.

For $j \in \mathbb{N}_0$ define

$$\Delta_j = \{\sigma \in \Delta^- \mid d(\sigma, U^+C^-) \leq j\}.$$

We show that

$$d(\sigma, U^+C^-) = 0 \iff \sigma \text{ op } C^+.$$

For $g \in U^+$

$$\delta^*(C^+, gC^-) = \delta^*(gC^+, gC^-) = \delta^*(C^+, C^-) = 1_W$$

using that U^+ fixes C^+ and that the Weyl-distance is G-invariant. Thus, using e.g. [AB08, Corollary 5.141], since $\delta^-(\sigma, gC^-) = 1_W = \delta^*(C^+, gC^-)$, we can conclude σ is opposite C^+. Hence we have $\Delta_0 = \Delta^-_{\mathrm{op}}$.

Let $X_j = |\Delta_j|$. Then $X = \cup_{j\geq 0} X_j$.

In [Abr96] it is shown that in this set-up, the conditions of [Abr96, Lemma 14] are satisfied, in particular

(i) X is n-connected,

(ii) $X_{j+1} = X_j \cup \bigcup_{i\in I_j} S_{i,j}$ where $S_{i,j}$ are contractible subspaces of X_{j+1}, I_j is a finite index set, satisfying

   (i) $S_{h,j} \cap S_{i,j} \subseteq X_j$ for all j and all $h \neq i \in I_j$.

   (ii) $S_{i,j} \cap X_j$ is $(n-2)$-connected.

Using the Mayer-Vietoris theorem (see e.g. [Hat02, Section 2.2]), we get the exact sequence of $\mathbb{Z}$-homology (note the we omit $\mathbb{Z}$ from the notation in this proof)

$$\ldots \to H_i(X_0 \cap S_{1,0}) \to H_i(X_0) \oplus H_i(S_{1,0}) \to H_i(X_0 \cup S_{1,0}) \to H_{i-1}(X_0 \cap S_{1,0}) \to \ldots$$

Since $H_i(X_0 \cap S_{1,0}) = 0$ for all $0 \leq i \leq n-2$ and $H_i(S_{1,0}) = 0$ for all i, we have

$$H_i(X_0) \cong H_i(X_0 \cup S_{1,0}).$$

We show by induction on k that $H_i(X_0) \cong H_i(X_0 \cup \bigcup_{i=1}^k S_{i,0})$. We have already covered $k = 1$. Note that

$$(X_0 \cup \bigcup_{i=1}^{k} S_{i,0}) \cap S_{k+1,0} = X_0 \cap S_{k+1,0} \cup \bigcup_{i=1}^{k} S_{i,0} \cap S_{k+1,0} = X_0 \cap S_{k+1,0}$$

since $S_{i,0} \cap S_{k+1,0} \subseteq X_0$. In particular, this set is again $(n-2)$-connected.

Thus, analogous to above $H_i(X_0 \cup \bigcup_{i=1}^k S_{i,0}) \cong H_i(X_0 \cup \bigcup_{i=1}^{k+1} S_{i,0})$. Hence $H_i(X_0) \cong H_i(X_0 \cup \bigcup_{i=1}^m S_{i,0}) = H_i(X_1)$, and proceeding inductively, we get $H_i(X_0) \cong H_i(X_j)$ for all $0 \leq i \leq n-2, j \geq 0$.

By e.g. [Hat02, Proposition 3.33], we have $H_i(X)$ is the direct limit of the $H_i(X_j)$. Since the $H_i(X_j) \cong H_i(X_0)$ for all j we get $H_i(X) \cong H_i(X_0)$ for all $0 \leq i \leq n-2$.

Together with the fact that X_0 is simply connected, the Hurewicz theorem implies that X_0 is $(n-2)$-connected and thus so is Δ^-_{op}. □

We now state and prove a high-dimensional version of Theorem 7.1.1 under the assumption that N acts freely on X. Both the statement and the proof rely on the theory of group cohomology. We will not recall the basics here, but refer to [Bro82]. We always consider the case of trivial action on the coefficient group. This result is not new, it has been stated in slightly different formulations before, e.g. in [CE56, Chapter 9, Application 1]. Afterwards, we will prove that in the KMS complex setting the action of N is indeed free.

Theorem 7.1.2. Let X be a connected pure n-dimensional simplicial complex, N be a group acting freely on X, and $\mathbb{A}$ an abelian coefficient group. Assume $p : X \to N\backslash X$ is rigid. If $H^j(X, \mathbb{A}) = 0$ and $H^j(N, \mathbb{A}) = 0$ for all $1 \leq j \leq k$ then $H^j(N\backslash X, \mathbb{A}) = 0$ for all $1 \leq j \leq k$.

Proof. This follows from the Cartan-Leray spectral sequence for equivariant cohomology [Bro82, Chapter VII, Theorem 7.9] which yields

$$H^p(N, H^q(X, \mathbb{A})) \Rightarrow H^{p+q}(N\backslash X, \mathbb{A}).$$

We have to be a bit careful with the definition of $H^0(X, \mathbb{A})$ here. In the previous results, we used the reduced cohomology. In particular, if X is connected then the reduced cohomology is zero. For this result we need to consider the non-reduced cohomology (the only difference is in degree 0). The only thing we need to know here is that if X is connected then $H^0(X, \mathbb{A}) = \mathbb{A}$ (see e.g. [Hat02, Proposition 2.7]).

We denote

$$E^{p,q} = H^p(N, H^q(X, \mathbb{A})).$$

Note that $E^{i,0} = H^i(N, \mathbb{A}) = 0$, and since $H^q(X, \mathbb{A}) = 0$ for all $1 \leq q \leq k$ we have $E^{p,q} = H^p(N, 0) = 0$ for all $p \geq 0$ and all $1 \leq q \leq k$.

The spectral sequence gives rise to a family of short exact sequences. For $p + q = 1$:

$$0 \to \underbrace{E^{0,1}}_{=0} \to H^1(N\backslash X, \mathbb{A}) \to \underbrace{E^{1,0}}_{=0} \to 0$$

For $p + q = i$ we have i many short exact sequences

$$\begin{gathered} 0 \to \underbrace{E^{0,i}}_{=0} \to F_1 A_i \to \underbrace{E^{1,i-1}}_{=0} \to 0 \\ 0 \to F_1 A_i \to F_2 A_i \to \underbrace{E^{2,i-2}}_{=0} \to 0 \\ \vdots \\ 0 \to F_{i-1} A_i \to H^i(N\backslash X, \mathbb{A}) \to \underbrace{E^{i,0}}_{=0} \to 0. \end{gathered}$$

Starting from the first row, we can deduce that all involved terms are 0. Thus in particular $H^i(N\backslash X, \mathbb{A}) = 0$ for all $1 \leq i \leq k$. $\square$

Lemma 7.1.3. Let A be a rank $n + 1$, n-spherical GCM over finite field K with $|K| \geq 4$ and let $\mathcal{U}_A(K)$ be the associated KMS group. Let $\phi : \mathcal{U}_A(K) \to G$ be a finite quotient giving rise to a KMS complex. Let $N = \ker \phi$. Then N acts freely on the fundamental KMS complex $\mathcal{CC}(\mathcal{U}_A(K); (I_{I\backslash\{i\}})_{i\in I})$.

Proof. Note that the action of $\mathcal{U}_A(K)$ and hence also of N is type preserving. Furthermore, we know that each vertex stabilizer in $\mathcal{U}_A(K)$ is a conjugate of one of the local groups $U_{I\backslash\{i\}}$. Since N is a normal subgroup of $\mathcal{U}_A(K)$, we can deduce that the action of N is free if and only if $N \cap U_{I\backslash\{1\}} = \{1\}$ for all $i \in I$. But the later holds, since ϕ is injective on the local groups. $\square$

Question. Let A be a rank $n+1$, n-classical GCM over a large enough finite field K and $\mathcal{U}_A(K)$ be the associated KMS group. Let $\phi_j : \mathcal{U}_A(K) \to G_j$ be a family of finite quotients of the KMS group giving rise to KMS complexes, i.e. satisfying the assumptions of Definition 4.2.2. Set $N_j = \ker(\phi_j)$. Can we find a family of finite quotients ϕ_j such that $H^1(N_j, \mathbb{F}_2) = 0$ and $H^2(N_j, \mathbb{F}_2) = 0$ for all j?

In particular, can we show $H^2(N_j, \mathbb{F}_2) = 0$ for the kernels of the congruence KMS complexes as in Section 6.4?

Using the discussion above, a positive answer to the question would yield the first examples of 2-coboundary expanders.

7.2 Generalization to other groups

In this section, we attempt to extract the key properties of KMS groups that we need to prove the high-dimensional expansion results. In particular, we are looking for the following objects.

- A group G acting on an infinite, locally finite, n-dimensional, $n+1$-partite simplicial complex X type-preservingly, transitively on the maximal faces and with finite vertex stabilizers.
- An infinite family of finite quotients $\phi_i : G \to G_i$ such that the quotient maps are injective on all vertex stabilizers G_v and such that $\phi_i(G_v) \cap \phi_i(G_w) = \phi_i(G_v \cap G_w)$. This would for example be satisfied if G is residually finite.
- A good description of all proper links in X. In particular, they need to be highly connected.
- To also get coboundary expansion, high connectivity of X is necessary.

Other direct limits We start by having a closer look at the spectral expansion results. In this case, only the links of codimension 2 faces need to be good spectral expanders and we can generalize our results to the following.

Theorem 7.2.1. Let I be a finite set and for each $J \subset I, 1 \leq |J| \leq 2$ let U_J be a finite group, together with inclusions $U_{\{i\}} \hookrightarrow U_{\{i,j\}}$ for all $i, j \in I$. Furthermore, we require that $U_{\{i,j\}} = \langle U_{\{i\}}, U_{\{j\}} \rangle$ for all $i, j \in I$. Let U be the direct limit of the $U_J, J \subset I, 1 \leq |J| \leq 2$ with respect to the inclusions, i.e. $U = *_{J \subset I, 1 \leq |J| \leq 2} U_J / (U_{\{i\}} \hookrightarrow U_{\{i,j\}})$. We require that

(i) there exists a $0 < \lambda \leq \frac{1}{|I|-1}$ such that $\mathcal{CC}(U_{\{i,j\}}; (U_{\{i\}}, U_{\{j\}}))$ is a λ-expander graph for all $i \neq j \in I$;

(ii) the groups $U_{\{i,j\}}$ must embed into U, i.e. the natural map from $U_{\{i,j\}}$ to U must be injective. Then we identify $U_{\{i\}}, U_{\{i,j\}}$ with their images in U.

Set $U_J := \langle U_{\{j\}} \mid j \in J \rangle \leq U$ for any $J \subsetneq I$. We further require that

(iii) $|U_J| < \infty$ for all $J \subsetneq I$;

(iv) $U_J \cap U_K = U_{J \cap K}$ for all $J, K \subsetneq I$.

Let G be a finite group such that there exists a surjective homomorphism $\phi : U \to G$ that is injective on all the U_J and such that

$$\phi(U_J) \cap \phi(U_K) = \phi(U_J \cap U_K) \text{ for all } J, K \subsetneq I.$$

Set $H_i = \phi(U_{I \setminus \{i\}})$, then $\mathcal{CC}(G; (H_i)_{i \in I})$ is a γ-spectral high-dimensional expander, for some $\gamma > 0$ independent of G and ϕ. An infinite family of such groups G and maps ϕ, where the size of the groups tends to infinity, gives rise to an infinite family of bounded degree HDX.

Proving this theorem works step by step exactly like the proof of Theorem 4.4.3, since we assumed exactly those properties of KMS-groups that we needed.

Question. Do other groups, satisfying the assumptions of Theorem 7.2.1 with infinitely many finite quotients exist?

Up to this point, we can only provide potential candidates. For $|I| = 3$, there exist other groups $U_{\{i\}}, U_{\{i,j\}}$ that satisfy the conditions of Theorem 7.2.1, for example the ones constructed in [LMW19, Theorem 4.6] or [Ron09, Chapter 4, Example 1]. In both examples, the difficulty is to find appropriate families of finite quotients which would be necessary to obtain an infinite family of bounded degree local spectral expanders.

Affine Kac–Moody groups Next, we consider a direction to generalize the idea of the congruence KMS complexes.

One option is considering the Phan geometry of an affine building, which can be thought of as a twisted version of the opposite complex. Their connectivity properties were studied in [DGM09; KW13], where the authors proved similar results as for the opposite complexes by adapting the techniques from [Abr96]. In this case, a twisted version of $\mathrm{Chev}(K[t, t^{-1}])$ acts on the complex and a similar trick as for the congruence KMS complexes can be used to get finite quotients.

We expect that a similar proof as for the KMS complexes should work without bigger challenges and would lead to new examples of both local-spectral and cosystolic high-dimensional expanders.

The second idea is to consider the affine Kac–Moody groups and their action on the so called opposition complex.

Definition 7.2.2. Given a twin building $(\Delta^+, \Delta^-, \delta^*)$ its *opposition complex* is given by pairs of opposite simplices:

$$\mathrm{Opp}(\Delta^\pm) = \{(\sigma, \tau) \mid \sigma \in \Delta^+, \tau \in \Delta^-, \mathrm{type}(\sigma) = \mathrm{type}(\tau), \sigma \text{ opposite } \tau\}.$$

Let A be an affine GCM and K a finite field. Then $G_A(K)$ acts on the opposition complex of the corresponding twin building and the action is transitive on the maximal simplices. Links inside opposition complexes are again opposition complexes of the corresponding subtype. Hence in the affine case, any proper link is the opposition complex of a spherical building. From [Hey03], we know that the opposition complexes of spherical buildings are n-connected. Quantifying this result to define cone functions for the spherical opposition complexes, similar to how we proceeded with Abramenko's work for opposite complexes, should be possible. On the other hand, affine Kac–Moody groups over finite fields are residually finite. Thus we get infinite families of finite quotients preserving the local structure. In particular, quotients similar to the congruence KMS complexes, should be possible. Note that non-affine Kac–Moody groups are not residually finite.

7.3 Generalization of coboundary and cosystolic expansion

The idea behind defining the coboundary and cosystolic expansion constant can be applied to arbitrary normed cochain and chain complexes.

Definition 7.3.1. (i) A chain complex (A_*, ∂_*) is a sequence of abelian groups $A_i, i \in \mathbb{Z}$ together with morphisms $\partial_n : A_n \to A_{n-1}$ called boundary operators such that the composition of two consecutive boundary operators is zero, i.e. $\partial_{n-1} \circ \partial_n = 0$ for all n.

(ii) A cochain complex (A^*, d_*) is a sequence of abelian groups $A^i, i \in \mathbb{Z}$ together with morphisms $d_n : A^n \to A^{n+1}$ called coboundary operators such that the composition of two consecutive coboundary operators is zero, i.e. $d_{n+1} \circ d_n = 0$ for all n.

(iii) A norm on an abelian group A is a map $\|\cdot\|: A \to \mathbb{R}_{\geq 0}$ such that

- $\|a\|= 0 \iff a = 0$,
- for all $a \in A$ we have $\|-a\|= \|a\|$,
- for all $a, b \in A$ we have $\|a + b\|\leq \|a\|+\|b\|$.

(iv) A normed chain complex is a chain complex (A_*, d_*) such that each A_n is a normed group and all ∂_n are bounded, i.e. there exists $\lambda_n \in \mathbb{R}_{\geq 0}$ such that for all $a \in A_n$ we have $\|\partial_n(a)\|\leq \lambda_n\|a\|$.

(v) A normed cochain complex is defined analogously.

Definition 7.3.2. Let (C_*, ∂_*) be a normed chain complex and let $\lambda \in [0, \infty]$. We say that C_* satisfies the (λ, n)-*uniform boundary condition* (UBC) if for all $b \in B_n := \operatorname{Im} \partial_{n+1}$ there exists a $c \in C_{n+1}$ such that

$$\partial_n c = b \text{ and } \|c\|\leq \lambda\|b\|.$$

Dually, we say that a normed cochain complex (C^*, d_*) satisfies the (λ, n)-*uniform coboundary condition* (UCC) for some $\lambda \in [0, \infty]$ if for all $b \in B^n := \operatorname{Im} d_{n-1}$ there exists $c \in C^{n-1}$ such that

$$d_{n-1}c = b \text{ and } \|c\|\leq \lambda\|b\|.$$

Remark 7.3.3. The term "uniform boundary condition" appeared in [MM85] and other works e.g. [FL21], where a chain complex is said to satisfy the n-uniform boundary condition if there exists a $\lambda \in [0, \infty)$ such that the chain complex satisfies (λ, n)-UBC.

The uniform coboundary condition appeared in [FLM23; MN23] in the following specific case. Here $C^n = C_b^n(\Gamma, \mathbb{R})$ is the cochain complex of bounded cohomology of the group Γ with ∞-norm. They further assume that $H_b^n(\Gamma, \mathbb{R}) = 0$. Then the *$n$-vanishing modulus* of Γ is the minimum over all λ such that $C_b^*(\Gamma, \mathbb{R})$ satisfies (λ, n)-UCC.

But also cosystolic and coboundary expansion fits into this framework. For $\mathbb{F}_2$ coefficients, a similar statement was shown in [KKL16] under the name of cofilling constant.

Proposition 7.3.4. Let X be a pure n-dimensional, finite simplicial complex and $\mathbb{A}$ an abelian group. Then $h_{cs}^{k-1}(X, \mathbb{A}) \geq \lambda > 0$ if and only if $C^* = C^*(X, \mathbb{A})$ with norm induced by the Garland weights satisfies $(\frac{1}{\lambda}, k)$-UCC.

Proof. Start by assuming $h_{cs}^{k-1}(X, \mathbb{A}) \geq \lambda > 0$. Let $b \in B^k(X, \mathbb{A})$. If $\|b\|= 0$ then $b = 0$ and we can choose $c = 0$. So assume $\|b\|\neq 0$ and let $f \in C^{k-1}(X, \mathbb{A}) \setminus Z^{k-1}(X, \mathbb{A})$ such that $df = b$ (which always exists since b is in the image of d). Thus,

$$\lambda \leq \frac{\|df\|}{\min_{z\in Z^{k-1}}\|f - z\|} \iff \min_{z\in Z^{k-1}}\|f - z\|\leq \frac{1}{\lambda}\|df\|.$$

Now, we choose c such that $\|c\|= \min_{z\in Z^{k-1}}\|f - z\|$ which satisfies $dc = df = b$. Together we get

$$\|c\|\leq \frac{1}{\lambda}\|df\|= \frac{1}{\lambda}\|b\|$$

as required.

Conversely, assume that C^* satisfies $(\frac{1}{\lambda}, k)$-UCC. Let $f \in C^{k-1} \setminus Z^{k-1}$, in particular $\|df\|, \|f\|\neq 0$. By UCC there exists $c \in C^{k-1}$ such that $dc = df$ and

$$\|c\|\leq \frac{1}{\lambda}\|df\|.$$

Note that $f - c \in Z^{k-1}$ since $df = dc$. Thus

$$\frac{\|df\|}{\min_{z\in Z^{k-1}}\|f - z\|} \geq \frac{\|df\|}{\|c\|} \geq \lambda$$

which shows $h_{cs}^{k-1}(X, \mathbb{A}) \geq \lambda$. □

Corollary 7.3.5. A finite, pure n-dimensional simplicial complex X is a $(\lambda, \mathbb{A})$-coboundary expander if and only if $H^i(X, \mathbb{A}) = 0$ for all $0 \leq i \leq n-1$ and $C^*(X, \mathbb{A})$ satisfies (λ, i)-UCC for all $0 \leq i \leq n - 1$.

Remark 7.3.6. Note that while every finite pure simplicial complex has a positive cosystolic expansion constant, not every chain complex satisfies UBC for some a finite λ. But for example if M is $(n - 1)$-connected then $C_*(M, \mathbb{Z})$ satisfies (λ, n)-UBC for some finite λ, see [FL21, Remark 9.2].

In the case of the vanishing modulus we have a similar result which roughly states that if $H_b^n(\Gamma, E) = 0$ then $C_b^*(\Gamma, E)$ satisfies (λ, n)-UCC for some finite λ. See [Cam+25, Lemma 2.23] for more details.

Remark 7.3.7. Some interesting results about UBC are obtained in [MM85] where they show that for a topological space M and $n \in \mathbb{N}$ the following are equivalent

(i) The normed chain complex $C_*(M, \mathbb{R})$ satisfies (λ, n)-UBC for some $\lambda < \infty$.

(ii) The homomorphism $H_b^{n+1}(M, \mathbb{R}) \to H^{n+1}(M, \mathbb{R})$ induced by the canonical inclusion $C_b^{n+1}(M, \mathbb{R}) \to C^{n+1}(M, \mathbb{R})$ is injective.

It was then shown that if M is a topological space with amenable fundamental group, then for any $n \in \mathbb{N}$ the chain complex $C_*(M, \mathbb{R})$ satisfies (λ, n)-UBC for some $\lambda < \infty$.

It is also interesting to note that in the context of UBC for topological spaces arguments similar to the cone functions approach appear, for example in the proof of [FL21, Lemma 4.1] and in [FLM23, Example 4.11].

Question. What can be said about UBC and UCC for general normed (co)chain complexes? Can techniques from one viewpoint be transferred to the other?

Some small results about UBC for arbitrary normed chain complexes are collected in [LLM26, Appendix A].

Dutch summary

Expander grafen zijn ijl maar tegelijk zeer sterk samenhangende grafen. Deze eigenschappen lijken tegenstrijdig, maar het bestaan van expander grafen kan worden aangetoond met een eenvoudige probabilistisch argument en inmiddels zijn er veel expliciete constructies bekend. Expander grafen zijn bijzonder nuttig in diverse toepassingen binnen de informatica en wiskunde. Daarom begon de expander-gemeenschap in het begin van de jaren 2000 met het veralgemenen van wat "sterk samenhangend" betekent naar simpliciale complexen als analogen van grafen in hogere dimensie. Helaas bestaat er niet één canonieke manier om het concept van expansie uit te breiden, aangezien verschillende – voor grafen equivalente – karakterisaties leiden tot verschillende versies van hoogdimensionale expansie. In dit werk richten wij ons op lokaal spectrale, coboundary en cosystolische expansie. Hoogdimensionale expanders hebben hun nut al bewezen in verschillende toepassingen, bijvoorbeeld bij de constructie van goede property-testers en kwantum LDPC-codes, bij vragen rond het mengen van Markov-ketens, en zij komen voor in overwegingen over het bestaan van niet-sofic groepen. In tegenstelling tot het geval van grafen zijn er slechts zeer weinig constructies van hoogdimensionale expanders bekend.

In deze thesis construeren wij een nieuwe familie van complexen, de KMS complexen, die nieuwe voorbeelden leveren van lokaal spectrale, coboundary- en cosystolische expanders. De constructie is gebaseerd op de rijke structuur van Kac–Moody–Steinberg (KMS)-groepen, die geamalgameerde producten zijn van maximale unipotente deelgroepen van Chevalley groepen. De KMS groepen kunnen surjectief afgebeeld worden op de maximale unipotente deelgroep van de overeenkomstige minimale Kac–Moody groep, en in veel gevallen zijn ze isomorf. In het bijzonder bezitten KMS groepen veel eindige quotiënten. Voor elk voldoende groot dergelijk quotiënt construeren wij een simpliciaal complex als een nevenklassencomplex over het quotiënt ten opzichte van de maximale lokale deelgroepen van de KMS groep. De KMS complexen hebben de eigenschap dat de lokale structuur alleen afhangt van de KMS groep en niet van het eindige quotiënt. De lokale structuur, de niet-triviale schakels, zijn elk isomorf aan het complex tegengesteld aan een kamer in een sferisch gebouw (spherical building). We kunnen de rijke structuur die deze tegenovergestelde complexen van de gebouwen erven, gebruiken om lokaal spectrale expansie voor alle KMS complexen aan te tonen. Verder tonen we aan dat de tegenovergestelde complexen van klassieke (d.w.z. type A_n, C_n, D_n) gebouwen coboundary-expanders zijn. Om dit te bereiken gebruiken we de technieken van cone-functies en ontwikkelen we nieuwe methoden om cone-functies inductief te construeren. Vervolgens leiden we hieruit cosystolische expansie af voor de klassieke KMS complexen.

We onderzoeken verder een specifieke klasse van KMS complexen, de congruentie KMS complexen, die voortkomen uit expliciete quotiënten binnen Chevalley groepen. Voor deze klasse kunnen we aantonen dat de eerste cohomologiegroep verdwijnt en daardoor 1-coboundary expansie afleiden voor de meeste abelse en niet-abelse coëfficiëntengroepen. Ten slotte bespreken we hoe deze ideeën kunnen worden veralgemeend naar andere groepen en hoe KMS complexen mogelijk de eerste voorbeelden van 3-dimensionale coboundary expanders zouden kunnen vormen.

Dit hoofdstuk is vertaald met de AI-assistentie van [Lum26].

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