High Dimensional Expanders

form Kac-Moody-Steinberg Groups

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Combinatorics

Simplical Complex

Definition (Simplical complex).
V set of vertices
$X \subseteq \mathcal{P}(V)$ with

- $\{v\} \in X, \forall v \in V$
- $\tau \in X, \sigma \subseteq \tau \Rightarrow \sigma \in X$

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Further notation:

- $\tau \in X : \dim(\tau) = |\tau| - 1$
- $X(k) = \{\tau \in X \mid \dim(\tau) = k\}$
- X is pure, d-dimensional if $\forall \tau \in X \; \exists \sigma \in X(d) : \tau \subseteq \sigma$

Links in a Simplical Complex

Definition (Link).
Let $\sigma \in X$, then

$$\mathrm{lk}_X(\sigma) = \{\tau \in X \mid \tau \cap \sigma = \emptyset, \tau \cup \sigma \in X\}$$

If $\sigma \in X(d-2)$ then $\mathrm{lk}_X(\sigma)$ is a graph.

Random Walk Matrix and its Spectrum

Definition.
For a (finite, simple) graph $G = (V, E)$, its random walk matrix $M \in \mathrm{Mat}_{|V|}(\mathbb{R})$ is given by, for $v, w \in V$

$$M_{v,w} = \begin{cases} \frac{1}{\deg(v)} & \text{if } \{v,w\} \in E \\ 0 & \text{if } \{v,w\} \notin E \end{cases}$$

M has eigenvalues

$$1 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{|V|}$$

High Dimensional Expanders

Definition (Oppenheim 2018).
Let X be a pure, d-dimensional simplicial complex satisfying:

- X is connected;
- $\mathrm{lk}_X(\sigma)$ is connected for all $\sigma \in X(i)$ for $i \leq d-2$;
- $\lambda_2(\mathrm{lk}_X(\sigma)) \leq \gamma \leq \frac{1}{d}$ for all $\sigma \in X(d-2)$.

Then X is a $\left(\frac{\gamma}{1-(d-1)\gamma}\right)$– (local spectral) HDX.

High Dimensional Expanders - Remarks

- Other definitions of HDX exist $\rightarrow$ not equivalent!
- local spectral expansion $\Rightarrow$ fast convergence of random walk
- applications in computer science and pure math

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Known constructions so far

- Lubotzky, Samuels, and Vishne 2005: Ramanujan complexes using quotients of Bruhat-Tits buildings
- Kaufman, Oppenheim 2018: spectral HDX using coset complexes of elementary matrices
- O'Donnell, Pratt 2022: spectral HDX using coset complexes for many Chevalley groups

Groups

Simplical Complexes from Groups

Definition (Coset complex).
Let G group, $H_0, \ldots, H_d \leq G$ subgroups, then $\mathcal{CC}(G; H_0, \ldots, H_d)$ is a pure, d-dimensional SC with

- vertices $\bigsqcup_{i=0}^{d} G/H_i$
- $\{g_1 H_{i_1}, \ldots, g_k H_{i_k}\}$ forms a $(k-1)$-simplex iff $\bigcap_{j=1}^{k} g_j H_{i_j} \neq \emptyset$

Maximal dimensional simplices: $\{gH_0, \ldots, gH_d\}$ for $g \in G$.

Dynkin Diagrams and Root Subgroups

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Fix a finite field k, $|k| \geq 4$, then for each $i \in I$ we have a **root subgroup** $U_i \cong (k, +)$

Kac-Moody-Steinberg Group

Given a Dynkin diagram A with vertex set I, we define for each $J \subsetneq I$ a group

$$U_J = \langle U_i, i \in J \mid \mathcal{R} \rangle$$

where the relations $\mathcal{R}$ give conditions for the commutators $[U_i, U_j]$ depending on the number of edges that connect i and j in A.

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Definition (Kac-Moody-Steinberg Group).

$$\mathcal{U}_A(k) := *_{J \subsetneq I} U_J / (U_J \hookrightarrow U_K, J \subseteq K \subsetneq I)$$

Our Results: New High Dimensional Expanders

The Main Theorem - Ingredients

Ingredients: finite group G, finite field k with $|k| = p^m$, a Dynkin diagram A like before s.t. $|I| = d + 1$, KMS-group $\mathcal{U}_A(k)$

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$\phi : \mathcal{U}_A(k) \to G$ homomorphism satisfying

- ϕ is injective on U_J for each $J \subsetneq I$
- for each $J, K \subsetneq I$, we have

$$\phi(U_J) \cap \phi(U_K) = \phi(U_J \cap U_K)$$

- ϕ is surjective

The Main Theorem - Result

Theorem.
Set $H_i = \phi(U_{I\setminus\{i\}})$, then

$$\mathcal{CC}(G,(H_i)_{i\in I})$$

is a $\left(\frac{\frac{2}{\sqrt{p}}}{1-(d-1)\frac{2}{\sqrt{p}}}\right)$ – HDX.

Infinte Families

Definition (Infinite family of bounded degree HDX).
$(X_m)_{m\in\mathbb{N}}$ a family of pure, d-dim SC such that $\exists\lambda, \Delta$ satisfying that, for all $m \in \mathbb{N}$, we have

- $|X_m(0)| \to \infty$ as $m \to \infty$
- For all $\tau \in X_m$: $|\{\sigma \in X(d) \mid \tau \subseteq \sigma\}| \leq \Delta$
- X_m is a λ – HDX

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We found two ways to construct such infinite families using our main theorem:

- by using that $\mathcal{U}_A(k)$ is residually finite in many cases
- by looking at quotients of affine $\mathcal{U}_A(k)$ inside Chevalley groups

Infinite Family of finite quotients - Sl_3

$$\phi : \mathcal{U}_A(k) = \langle U_i \mid i = 0, 1, 2 \rangle \to Sl_3(k[t])$$

$$U_1(\lambda) \mapsto \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \phi(U_{02}) = \begin{pmatrix} 1 & 0 & 0 \\ kt & 1 & k \\ kt & 0 & 1 \end{pmatrix}$$

$$U_2(\lambda) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix} \qquad \phi(U_{01}) = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ kt & kt & 1 \end{pmatrix}$$

$$U_0(\lambda) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda t & 0 & 1 \end{pmatrix} \qquad \phi(U_{12}) = \begin{pmatrix} 1 & k & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

Fix a family of irreducible polynomials $(f_m)_{m \in \mathbb{N}}$ s.t. for all m:

$$\deg(f_m) \geq 2, \quad (t) + (f_m) = k[t], \quad \lim_{m \to \infty} \deg(f_m) = \infty$$

Define $\psi_m = \pi_m \circ \phi : \mathcal{U}_A(k) \to Sl_3(k[t]/(f_m))$ and set

$$G_m = Sl_3(k[t]/(f_m))$$

$$H_0^m = \pi_m(\phi(U_{12})),\ H_1^m = \pi_m(\phi(U_{02})),\ H_2^m = \pi_m(\phi(U_{01}))$$

Corollary.
$X_m = \mathcal{CC}(G_m; H_0^m, H_1^m, H_2^m), m \in \mathbb{N}$ is an infinite family of bounded degree HDX.