High-dimensional expansion in buildings and their opposite complexes

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Expander Graphs

Let $G = (V, E)$ be a finite, simple graph, $n = |V|$.

Definition (Cheeger constant).

$$h(G) = \min_{\emptyset \neq A \subset V} \frac{|E(A, V \setminus A)|}{\min\{|A|, |V \setminus A|\}}$$

where $E(A, V \setminus A) = \{\{v_1, v_2\} \in E \mid v_1 \in A, v_2 \notin A\}$.

G is a λ-expander if $h(G) \geq \lambda$.

Simplicial cohomology

Let X be a pure d-dimensional finite simplicial complex.

i-dimensional faces: $X(i) = \{\sigma \in X \mid |\sigma| = i+1\},\ X(-1) = \{\emptyset\}$

i-cochains: $C^i(X, \mathbb{F}_2) = \{f : X(i) \to \mathbb{F}_2\}$

i-th coboundary map

$$\delta_i : C^i(X, \mathbb{F}_2) \to C^{i+1}(X, \mathbb{F}_2) :\ \delta_i(f)(\sigma) = \sum_{\tau \in X(i) : \tau \subset \sigma} f(\tau)$$

cocycles	coboundaries	cohomology group
$Z^i(X, \mathbb{F}_2) = \ker \delta_i$	$B^i(X, \mathbb{F}_2) = \operatorname{im} \delta_{i-1}$	$H^i(X, \mathbb{F}_2) = Z^i(X, \mathbb{F}_2)/B^i(X, \mathbb{F}_2)$

Coboundary expansion

Let X be a pure d-dimensional finite simplicial complex. For $f \in C^i(X, \mathbb{F}_2)$ define

$$\|f\| = \sum_{\sigma \in \mathrm{supp}(f)} \frac{|\{\tau \in X(d) \mid \sigma \subseteq \tau\}|}{|X(d)|\binom{d+1}{i+1}}.$$

Coboundary expansion constant

$$h^i_{cb}(X, \mathbb{F}_2) = \min \left\{ \frac{\|\delta_i(f)\|}{\min_{b \in B^i(X, \mathbb{F}_2)} \|f + b\|} \mid f \in C^i(X, \mathbb{F}_2) \setminus B^i(X, \mathbb{F}_2) \right\}$$

X is an ε-**coboundary expander** if $h^i_{cb}(X, \mathbb{F}_2) \geq \varepsilon$ for all $0 \leq i \leq d-1$.

Infinite families

G connected $\iff h(G) > 0 \quad \rightsquigarrow \quad H^i(X, \mathbb{F}_2) = 0 \iff h^i_{cb}(X, \mathbb{F}_2) > 0$

Infinite family of bounded degree HDX

Let $\varepsilon > 0, c, d \in \mathbb{N}$ be fixed. A family $(X_s)_{s\in\mathbb{N}}$ of finite, pure, d-dimensional complexes is a family of coboundary expanders if

- for all $s \in \mathbb{N}$: X_s is an ε-coboundary expander
- for all $s \in \mathbb{N}, v \in X_s(0)$: $\deg(v) := |\{\tau \in X(d) \mid v \subseteq \tau\}| \leq c$
- $|X_s(0)| \to \infty$ for $s \to \infty$.

Cosystolic expansion

Problem: Showing $H^i(X, \mathbb{F}_2) = 0$ is hard. The only infinite families of coboundary expanders known are 2-dimensional.

Cosystolic expansion constant

$$h^i_{cs}(X, \mathbb{F}_2) = \min \left\{ \frac{\|\delta_i(f)\|}{\min_{z \in Z^i(X, \mathbb{F}_2)} \|f + z\|} \mid f \in C^i(X, \mathbb{F}_2) \setminus Z^i(X, \mathbb{F}_2) \right\}$$

X is a (ε, μ)-**cosystolic expander** if $h^i_{cs}(X, \mathbb{F}_2) \geq \varepsilon$ and $\|z\| \geq \mu$, $\forall z \in Z^i(X, \mathbb{F}_2) \setminus B^i(X, \mathbb{F}_2)$ for all $0 \leq i \leq d-1$.

Local-to-global

Theorem (Evra-Kaufman 2017).
Let X be a finite, pure, d-dimensional simplicial complex. For all $\beta > 0$ there exist $\alpha, \varepsilon, \mu > 0$ such that if

- X is an α-local spectral expander,
- all links $\mathrm{lk}(\tau)$ with $0 \leq \dim(\tau) \leq d-2$ are β-coboundary expander,

then the $(d-1)$-skeleton of X is an (ε, μ)-cosystolic expander.

Examples

Known examples of infinite families of cosystolic expanders in arbitrary dimension:

Ramanujan complexes (Lubotzky-Samuels-Vishne '05)
$\rightarrow$ links are spherical buildings

KMS complexes (Grave de Peralta-VB, Oppenheim-VB)
$\rightarrow$ links are complex opposite the fundamental chamber inside spherical building

More examples in dimension 2, e.g. Kaufman-Oppenheim, O'Donnel-Singer, even coboundary expanders.

Cone function

Let X be a d-dimensional simplicial complex, $d \geq 2$. A 1-cone function for X is a triple $\left(v_0, (P_u)_{u \in X(0)}, (T_{(u,w)})_{(u,w) \in \vec{X}(1)}\right)$ such that

- $v_0 \in X(0)$,
- $\forall u \in X(0) : P_u$ is a path from v_0 to u,
- $T_{(u,w)} = (P_0, \ldots, P_m)$ such that
  - $P_0 = P_u \circ (u, w) \circ P_w^{-1}$
  - $P_i \sim_{BT} P_{i+1}$ or $P_i \sim_{TR} P_{i+1}$,
  - $P_m = (v_0)$.

Note: If a complex is contractible, we can find a cone function.

Cone radius and expansion

Radius of cone function

$$\mathrm{Rad}_1(\mathrm{Cone}_X) = \max_{\vec{(u,w)} \in X(1)} |\{0 \le i \le m \mid P_i \sim_{TR} P_{i+1} \text{ in } T_{(u,w)}\}|$$

Theorem (Kaufman-Oppenheim '21)

For every finite pure d-dimensional simplicial complex, if

- The group $\mathrm{Aut}(X)$ acts transitively on $X(d)$,
- X admits a 1-cone function Cone_X,

then

$$h^1_{cb}(X, \mathbb{F}_2) \ge \frac{1}{\mathrm{Rad}_1(\mathrm{Cone}_X)\binom{d+1}{3}}.$$

Cone function for spherical building

Spherical building from my perspective:

- finite simplicial complex Δ with family of subcomplex $\mathcal{A}$ called apartments
- each $A \in \mathcal{A}$ is isomorphic to one fixed Coxeter complex $\rightarrow$ contractible if $\dim(A) \geq 2$
- for all $\sigma, \tau \in \Delta : \exists A \in \mathcal{A} : \sigma, \tau \in A$.

Cone for spherical building:

- choose a vertex $v_0 \in \Delta(0)$
- for each $\sigma \in \Delta$ set $B_\sigma = \bigcap_{A \in \mathcal{A} : \sigma, v_0 \in A} A$.
- Each B_σ is contractible. Can define Cone_Δ such that paths corresponding to σ lie in B_σ.
- Hence $\mathrm{Rad}(\mathrm{Cone}_\Delta) \leq \max_\sigma |B_\sigma| \Rightarrow$ only depends on type, not on thickness.

Opposite complex

Let Δ be a d-dimensional (thick) spherical building. There is an opposition relation on the chambers (maximal simplices).

Opposite complex
Fix a chamber $C \in \Delta(d)$.

$$\Delta^0(C) = \{\sigma \in \Delta \mid \exists D \in \Delta(d) : \sigma \subseteq D \text{ and } D \text{ opposite } C\}.$$

Theorem (OVB'25)
Let Δ be a thick spherical building of type A_n, C_n or D_n, $C \in \Delta(n-1)$ a chamber. Then $\Delta^0(C)$ is a λ-coboundary expander, where λ depends only on the type.

Joins and adding vertices

Proposition (OVB '25)
Let X, Y be finite simplicial complexes with cone functions $\mathrm{Cone}_X, \mathrm{Cone}_Y$. Then there exists a cone function Cone_{X*Y} with $\mathrm{Rad}(\mathrm{Cone}_{X*Y})$ bounded.
Moreover, for any vertex v: $\mathrm{Rad}(\mathrm{Cone}_{X*\{y\}}) \leq 1$.

Theorem (OVB'25)
Let X be an d-dimensional finite simplicial complex, X' a full subcomplex with cone function $\mathrm{Cone}_{X'}$. Let $W \subseteq X(0)$ such that

- $W \cap X' = \emptyset$,
- $\forall w_1, w_2 \in W : \{w_1, w_2\} \notin X$,
- $\forall w \in W : \mathrm{lk}_X(w) \cap X'$ has cone function $\mathrm{Cone}_{\mathrm{lk}_X(w)\cap X'}$.

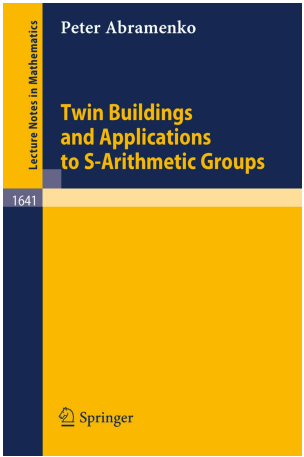
Then the full subcomplex spanned by $X'(0) \cup W$ has a cone function with bounded radius.

Cone for opposite complex

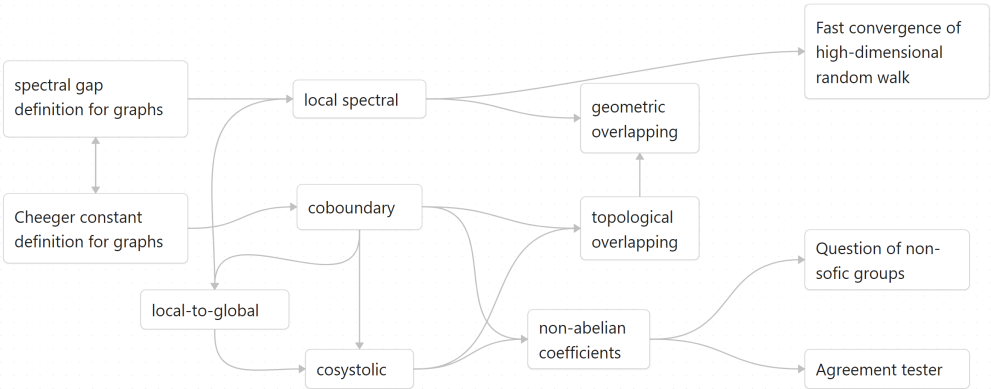
Abramenko shows that $\Delta^0(C)$ of dimension d is $(d-1)$-spherical.

He shows that $\Delta^0(C)$ can be constructed from a small contractible complex by adding vertices.

We go through the proof and keep track of the radius.



The world of HDX - a map



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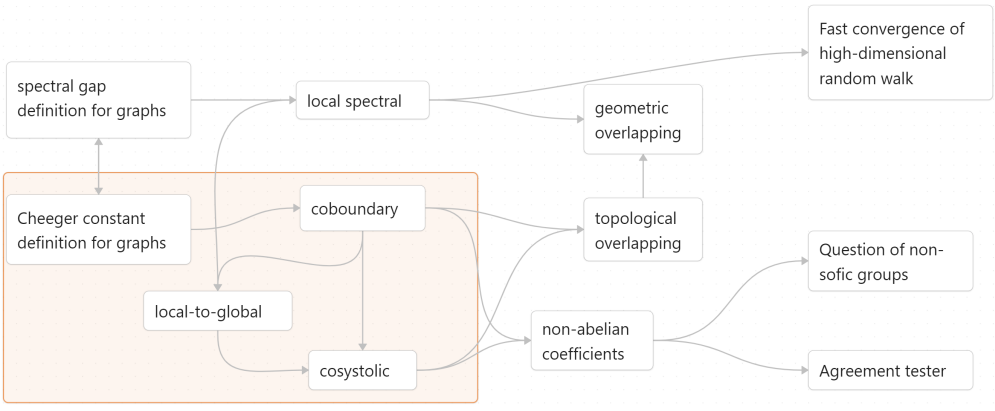

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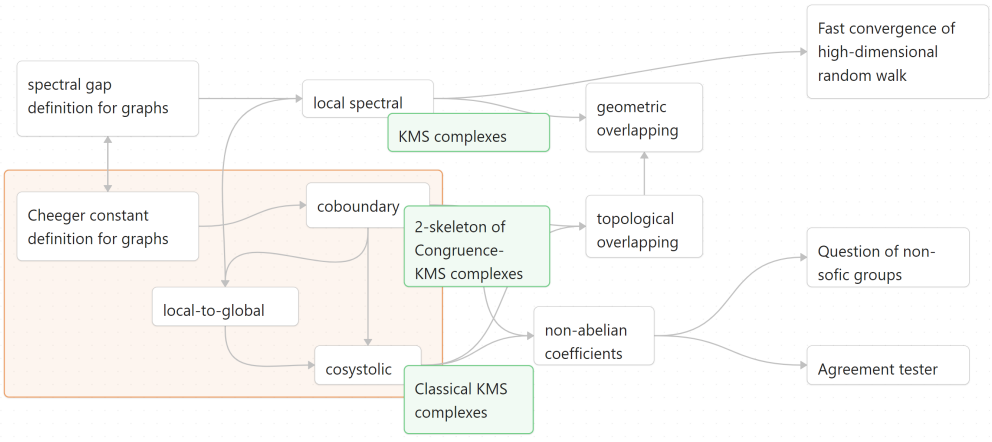

Infinite families via KMS complexes

KMS group

- $A = (A_{ij})_{i,j \in I}$ a GCM, rank n, $(n-1)$-spherical
- k a finite field
- for $J \subset I$: $U_J = U^+ \leq \mathrm{Chev}_{A_J}(k)$

$$\mathcal{U}_A(k) = \underset{J \subsetneq I}{\ast} U_J \Big/ (U_J \hookrightarrow U_L; J \subset L \subset I)$$

Coset complex

Let G be a group, $H_i \leq G, i \in I$.
$\mathcal{CC}(G; (H_i)_{i\in I})$ is the simplicial complex with

- vertices $\bigsqcup_{i\in I} G/H_i$
- maximal simplices $\{gH_i \mid i \in I\}, g \in G$.

KMS complex

Let $\varphi : \mathcal{U}_A(k) \to G$ be a finite quotient, satisfying some injectivity properties.

$$X_\varphi = \mathcal{CC}(G; (\varphi(U_{I\setminus\{i\}}))_{i\in I})$$

$\mathrm{lk}_X(\{gH_j \mid j \in J\}) \cong \mathcal{CC}(U_J; (U_{J\setminus\{i\}})_{i\in J}) \cong \Delta^0(C)$ where Δ is the spherical building corresponding to $\mathrm{Chev}_{A_J}(k)$.

Cone functions

Simplicial homology

i-chains: $C_i(X, \mathbb{F}_2) = \left\{ \sum_{\sigma \in X(i)} t_\sigma \sigma \mid t_\sigma \in \mathbb{F}_2 \right\}$

i-boundary map: $\partial_i : C_i(X, \mathbb{F}_2) \to C_{i-1}(X, \mathbb{F}_2) :$

$$\sum_{\sigma \in X(i)} t_\sigma \sigma \quad \mapsto \quad \sum_{\sigma \in X(i)} \sum_{\tau \in X(i-1) : \tau \subset \sigma} t_\sigma \tau$$

Cone functions

$$C_i(X, \mathbb{F}_2) = \left\{ \sum_{\sigma \in X(i)} t_\sigma \sigma \mid t_\sigma \in \mathbb{F}_2 \right\} \qquad \partial_i : C_i(X, \mathbb{F}_2) \to C_{i-1}(X, \mathbb{F}_2) : \sum_{\sigma \in X(i)} t_\sigma \sigma \mapsto \sum_{\sigma \in X(i)} \sum_{\tau \in X(i-1) : \tau \subset \sigma} t_\sigma \tau$$

Let X be a finite, d-dimensional complex, $v \in X(0)$.

An m-**cone function of** X **with apex** v is a collection of $\mathbb{Z}$-module homomorphisms $\mathrm{Cone}_X^i(X) : C_i(X) \to C_{i+1}(X)$ for $-1 \leq i \leq m$ such that

- $\mathrm{Cone}_X^{-1}(1 \cdot \emptyset) = 1 \cdot v$
- $\partial_{i+1} \mathrm{Cone}_X^i(A) + \mathrm{Cone}_X^{i-1}(\partial_i A) = A$ for all $0 \leq i \leq m, A \in C^i(X)$.

Set $\mathrm{Cone}_X = \bigoplus_{i=-1}^m \mathrm{Cone}_X^i$.

Cone radius and expansion

Radius of cone function

$\mathrm{Rad}(\mathrm{Cone}_X) = \max_k \max_{\sigma \in X(k)} |\mathrm{supp}(\mathrm{Cone}_X(1 \cdot \sigma))|$

Theorem (Kaufman-Oppenheim '21)

For every finite pure d-dimensional simplicial complex, if

- The group $\mathrm{Aut}(X)$ acts transitively on $X(d)$,
- X admits an m-cone function Cone_X,

then for every $0 \leq k \leq m$ it holds that

$$h^k_{cb}(X, \mathbb{F}_2) \geq \frac{1}{\mathrm{Rad}(\mathrm{Cone}_X)\binom{n+1}{k+1}}.$$

Examples

Let X be a 1-dimensional, connected simplicial complex.

1. Pick $v \in X(0)$ and set $\mathrm{Cone}_X(1\emptyset) = 1\{v\}$.
2. For every $w \in X(0) \setminus \{v\}$ fix a path $v = w_0, w_1, \cdots, w_n = w$,
3. set $\mathrm{Cone}_X(1\{w\}) = \sum_{i=0}^{n-1} 1\{w_i, w_{i+1}\}, \mathrm{Cone}_X(1\{v\}) = 0$.

Let X be 2-dimensional, connected and contractible.

1. Pick $v \in X(0)$.
2. Define $\mathrm{Cone}_X^{-1}, \mathrm{Cone}_X^0$ as above. Let P_w denote the path from v to w chosen in step 2 above.
3. For $\{u, w\} \in X(1)$, $P_w^{-1} \circ P_u$ is a loop starting in v through $\{u, w\}$. Let $\sigma_1, \ldots, \sigma_m$ be the triangles used for contracting the loop.
4. Set $\mathrm{Cone}_X(\{u, w\}) = \sum_{i=1}^{m} \sigma_i$.