Constructing High-Dimensional Expanders

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Expander Graphs

Definition.
A family of finite simple graphs $(G_m = (V_m, E_m))_{m\in\mathbb{N}}$ is called a **family of expanders** if there exist $d \in \mathbb{N}, \varepsilon > 0$ such that

1. $|V_m| \overset{m\to\infty}{\longrightarrow} \infty$,
2. G_m is d-regular for all $m \in \mathbb{N}$,
3. G_m is a ε-expander for all $m \in \mathbb{N}$.

The Graph Laplacian

Let $G = (V, E)$ be a finite, simple, graph, $n = |V|$.

- $L^2(V) = \{f : V \to \mathbb{R}\} \cong \mathbb{R}^n$
- inner product on $L^2(V)$: $\langle f, g\rangle = \sum_{v \in V} \deg(v) f(v) g(v)$

- For each edge $e \in E$ fix an orientation $e = (e^-, e^+)$
- define $\delta : L^2(V) \to L^2(E)$ via $(\delta f)(e) = f(e^+) - f(e^-)$.

- $\Delta = \delta^* \delta : L^2(V) \to L^2(V)$ is the Laplace operator of G.

$$(\Delta f)(v) = f(v) - \frac{1}{\deg(v)} \sum_{u:\{u,v\} \in E} f(u)$$

ε-Expander Graph

- Eigenvalues of Δ: $\lambda_1 = 0 \leq \lambda_2 \leq \cdots \leq \lambda_n$
- Let $L_0^2(V) = \{f \in L_2(V) \mid \langle f, 1 \rangle = 0\}$
- $$\lambda_2(G) = \inf \left\{ \frac{\langle \Delta f, f \rangle}{\langle f, f \rangle} \mid 0 \neq f \in L_0^2(V) \right\}$$

Definition.
Let $\varepsilon > 0$, then G is an ε-expander if

$$\lambda_2(G) \geq \varepsilon$$

Higher dimensions - cohomology

Let X be a pure d-dimensional finite simplicial complex.
Fix an order on the vertices $X(0) = \{v_1, ..., v_n\}, v_1 < \cdots < v_n$.
Let $\mathbb{F}$ be any abelian group.

Definition.

- collection of i-dimensional faces
  $X(i) = \{\sigma \in X \mid \dim(\sigma) = i\}, X(-1) := \{\emptyset\}$
- i-cochains of X: $C^i(X, \mathbb{F}) = \{f : X(i) \to \mathbb{F}\}$
- oriented incidence number for
  $\sigma = \{v_{j_0}, \ldots, v_{j_i}\} \in X(i), \tau \in X(i-1)$:

$$[\sigma : \tau] := \begin{cases} (-1)^\ell & \text{if } \sigma \setminus \tau = \{v_{j_\ell}\} \\ 0 & \text{if } \tau \not\subseteq \sigma \end{cases}$$

Definition.

i-th coboundary map, for $-1 \leq i \leq d-1$

$$\delta_i : C^i(X, \mathbb{F}) \to C^{i+1}(X, \mathbb{F}) : \; (\delta_i f)(\sigma) = \sum_{\tau \in X(i)} [\sigma : \tau] f(\tau)$$

- cocycles $Z^i(X, \mathbb{F}) = \ker \delta_i$
- coboundaries $B^i(X, \mathbb{F}) = \operatorname{im} \delta_{i-1}$
- i-th cohomology group $H^i(X, \mathbb{F}) = Z^i(X, \mathbb{F}) / B^i(X, \mathbb{F})$

Interesting cases: $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{F}_2$

For $f, g \in C^i(X, \mathbb{R})$ define

$$\langle f, g \rangle = \sum_{\sigma \in X(i)} \deg(\sigma) f(\sigma) g(\sigma)$$

where $\deg(\sigma) = |\{\tau \in X(d) \mid \sigma \subseteq \tau\}|$. This gives $C^i(X, \mathbb{R})$ a Hilbert space structure.

For $\mathbb{F} = \mathbb{F}_2$ we only define a norm on $C^i(X, \mathbb{F}_2)$:

$$\|f\| = \sum_{\sigma \in \mathrm{supp}(f)} \frac{\deg(\sigma)}{|X(d)|\binom{d+1}{i}}$$

Spectral expansion - $\mathbb{F} = \mathbb{R}$

Definition.
Let $\mathbb{F} = \mathbb{R}$, then for all $0 \leq i \leq d-1$ we define the higher order Laplacian of X as

$$\Delta_i^{\text{up}} = \delta_i^* \delta_i, \ \Delta_i^{\text{down}} = \delta_{i-1}\delta_{i-1}^*, \ \Delta_i = \Delta_i^{\text{up}} + \Delta_i^{\text{down}}$$

$$\lambda^{(i)}(X) = \inf_{f \in (B^i)^\perp} \frac{\langle \Delta_i f, f \rangle}{\langle f, f \rangle} = \inf_{f \in (B^i)^\perp} \left(\frac{\|\delta_i f\|}{\|f\|} \right)^2$$

X is an ε-spectral expander if $\lambda^{(i)}(X) \geq \varepsilon$ for all $0 \leq i \leq d-1$.

Coboundary expansion - $\mathbb{F} = \mathbb{F}_2$

We define two expansion parameters:

$$h_i(X) = \min\left\{ \frac{\|\delta_i(f)\|}{\min_{z\in Z^i}\|f+z\|} \mid f \in C^i \setminus Z^i \right\}$$

$$\mathrm{Syst}_i(X) = \min\{\|z\| \mid z \in Z^i \setminus B^i\}$$

X is an ε-**coboundary expander** if $h_i(X) \geq \varepsilon$ and $H^i(X, \mathbb{F}_2) = 0$ for all $0 \leq i \leq d-1$.

X is a (ε, μ)-**cosystolic expander** if $h_i(X) \geq \varepsilon$ and $\mathrm{Syst}_i(X) \geq \mu$ for all $0 \leq i \leq d-1$.

Local spectral expansion

Recall: $\mathrm{lk}_X(\tau) = \{\sigma \in X \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in X\}$

Definition.
A pure, finite, d-dimensional complex X is called a ε-**local spectral expander**, for some $\varepsilon > 0$, if

- the 1-skeleton of X is connected
- for every $\tau \in X$ with $\dim(\tau) \leq d - 2$ we have that the 1-skeleton of $\mathrm{lk}_X(\tau)$ is connected
- $\forall \tau \in X(d-2) : \lambda_2(\mathrm{lk}_X(\tau)) \geq \varepsilon$.

Important Theorems

Theorem (Oppenheim 2018).
If X is an ε-local spectral expander and $\varepsilon \geq 1 - \frac{1}{d}$ then X is an $\tilde{\varepsilon}$-spectral expander and an ε'-geometric expander.

Theorem (Evra-Kaufman 2017).
Let X be an α-local spectral expander such that all links $\mathrm{lk}_X(\tau)$ with $0 \leq \dim(\tau) \leq d - 2$ are β-coboundary expander. Then there exist $\varepsilon, \mu > 0$ such that the $(d-1)$-skeleton of X is a (ε, μ)-cosystolic expander.

Remarks

The complete simplex and spherical buildings are spectral and coboundary expander, but we want:

Infinite family of bounded degree HDX

Let $\varepsilon > 0, c, d \in \mathbb{N}$ be fixed. A family $(X_s)_{s\in\mathbb{N}}$ of finite, pure, d-dimensional complexes is a family of HDX if

- for all $s \in \mathbb{N}$: X_s is an ε-spectral/coboundary/cosystolic expander
- for all $s \in \mathbb{N}, v \in X_s(0)$: $\deg(v) \leq c$
- $|X_s(0)| \to \infty$ for $s \to \infty$.

Known HDX constructions

- First construction: Lubotzky, Samuels, Vishne (2005): quotients of Bruhat-Tits buildings by cocompact lattices $\rightarrow$ local spectral and cosystolic HDX
- Kaufman, Oppenheim (2018): coset complexes over $\mathrm{SL}_n(k[t]/(f)) \rightarrow$ local spectral expander
- O'Donnel, Pratt (2022): coset complexes over any Chevalley group $\rightarrow$ local spectral expander
- Grave de Peralta, V-B (2024): coset complexes over finite quotients of KMS groups $\rightarrow$ local spectral expander

Coset complex

Let G be a finite group and $H_0, \dots, H_d$ subgroups of G. Then the coset complex $\mathcal{CC}(G, (H_i)_{i=0}^d)$ is a d-dimensional simplical complex with

- vertices $\bigsqcup_{i=0}^d G/H_i$
- maximal faces $\{gH_0, \dots, gH_d\}$ for $g \in G$.

In particular $gH_i \sim hH_j$ if $i \neq j$ and $gH_i \cap hH_j \neq \emptyset$.

Example High-dimensional expander

- Fix a finite field k, $|k| \geq 4$, $\mathrm{char}(k) \neq 2$
- Consider the following subgroups inside $\mathrm{SL}_3(k[t])$:

$$H_0 = \begin{pmatrix} 1 & k & k \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}, H_1 = \begin{pmatrix} 1 & 0 & 0 \\ kt & 1 & k \\ kt & 0 & 1 \end{pmatrix}, H_2 = \begin{pmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ kt & kt & 1 \end{pmatrix}$$

- Fix irreducible polynomials $f_m \in k[t]$ with $\deg(f_m) \geq 2$ for $m \in \mathbb{N}$ such that $\deg(f_m) \to \infty$.
  $\pi_m : \mathrm{SL}_3(k[t]) \to \mathrm{SL}_3(k[t]/(f_m))$ the entry-wise projection.
- Set $G_m = \mathrm{SL}_3(k[t]/(f_m)), H_i^m = \pi_m(H_i), i = 0, 1, 2$

Then

$$(\mathcal{CC}(G_m, (H_0^m, H_1^m, H_2^m)))_{m \in \mathbb{N}}$$

is a family of bounded degree spectral expanders.

2 Generalization

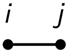
Let A be a Dynkin diagram with nodes I that is 2-spherical. Set, for all $i \in I$

$$U_i \cong (k, +)$$

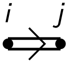
and for all $i \in I, i \neq j \in I$

if 

$: U_{ij} = U_i \times U_j$

if 

$$: U_{ij} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \mid a, b, c \in k \right\} \qquad U_i \leftrightarrow a, \;\; U_j \leftrightarrow c$$

if 

$$: U_{ij} = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ xz + a & z & 1 & 0 \\ c & a & -x & 1 \end{pmatrix} \mid x, z, a, c \in k \right\} \leq \mathrm{Sp}_4(k)$$

$U_i \leftrightarrow z, U_j \leftrightarrow x$

Kac-Moody-Steinberg (KMS) group

$$\mathcal{U}_A(k) = \mathop{\ast}_{i,j\in I} U_{\{i,j\}} \Big/ \left(U_i \xrightarrow{f_{i,j}} U_{ij};\, i \neq j \in I\right)$$
$$= \langle U_i, U_{ij}; i,j \in I \mid \forall i \neq j \in I, \forall a \in U_i : a = f_{i,j}(a)\rangle$$

- A not spherical $\Rightarrow \mathcal{U}_A(k)$ infinite and $U_{ij} \hookrightarrow \mathcal{U}_A(k)$
- For $J \subset I$ let $U_J = \langle U_j \mid j \in J\rangle \leq \mathcal{U}_A(k)$
  If sub-diagram induced by J is spherical, U_J is finite
- in many cases $\mathcal{U}_A(k)$ is residually finite
- if A is affine, $\mathcal{U}_A(k)$ has quotient inside Chevalley groups over $k[t]/(f)$ like in the example.

Main Theorem

- $\mathcal{U}_A(k)$ a KMS-group such that
  - k a finite field, $|k| \geq 4$
  - A a Dynkin diagram on nodes I such that any sub-diagram of size $|I| - 1$ is spherical
- G a finite group, $\phi : \mathcal{U}_A(k) \twoheadrightarrow G$ such that
  - $\phi|_{U_J}$ is injective for all $J \subsetneq I$
  - $\phi(U_J) \cap \phi(U_K) = \phi(U_J \cap U_K)$ for all $J, K \subsetneq I$.

Then

$$\mathcal{CC}\left(G, \left(\phi(U_{I\setminus\{i\}})\right)_{i\in I}\right)$$

is a γ-local spectral expander, where γ is independent of ϕ, G.

Open question and reference

Question

Are the constructed simplical complexes coboundary or cosystolic expanders?

[1] **A. Lubotzky. High dimensional expanders. *Proceedings of the International Congress of Mathematicians—Rio de Janeiro*, 2018**

[2] **T.Kaufman and I. Oppenheim. High dimensional expanders and coset geometries. *European Journal of Combinatorics*, 2023**

[3] **A. Lubotzky, B. Samuels and U. Vishne. Ramanujan complexes of type $\tilde{A}_d$. *Israel Journal of Mathematics*, 2005**

[4] **L. Grave de Peralta and I. Valentiner-Branth. High-dimensional expanders from Kac–Moody–Steinberg groups. *arXiv 2401.05197*, 2024**

Main Theorem - quick version

- $\mathcal{U}_A(k)$ a Kac-Moody-Steinberg group with
  - k a finite field, $|k| \geq 4$
  - A a Dynkin diagram on nodes I such that any sub-diagram of size $|I| - 1$ is spherical
- G a finite group, $\phi : \mathcal{U}_A(k) \twoheadrightarrow G$ with some injectivity properties.

Then

$$\mathcal{CC}\left(G, \left(\phi(U_{I\setminus\{i\}})\right)_{i\in I}\right)$$

is a γ-local spectral expander, where γ is independent of ϕ, G.

Question

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