New cosystolic high-dimensional expanders from KMS groups

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Abstract

Cosystolic expansion is a high-dimensional generalization of the Cheeger constant for simplicial complexes. Originally, this notion was motivated by the fact that it implies the topological overlapping property, but more recently it was shown to be connected to problems in theoretical computer science such as list agreement expansion and agreement expansion in the low soundness regime.

There are only a few constructions of high-dimensional cosystolic expanders and, in dimension larger than 2, the only known constructions prior to our work were (co-dimension 1)-skeletons of quotients of affine buildings. In this paper, we give the first coset complex construction of cosystolic expanders for an arbitrary dimension. Our construction is more symmetric and arguably more elementary than the previous constructions relying on quotients of affine buildings.

The coset complexes we consider arise from finite quotients of Kac–Moody–Steinberg (KMS) groups and are known as KMS complexes. KMS complexes were introduced in recent work by Grave de Peralta and Valentiner-Branth where it was shown that they are local-spectral expanders. Our result is that KMS complexes, satisfying some minor condition, give rise to infinite families of bounded degree cosystolic expanders of arbitrary dimension and for any finitely generated Abelian coefficient group.

This result is achieved by observing that proper links of KMS complexes are joins of opposition complexes in spherical buildings. In order to show that these opposition complexes are coboundary expanders, we develop a new method for constructing cone functions by iteratively adding sets of vertices. Hence we show that the links of KMS complexes are coboundary expanders. Using the prior local-to-global results, we obtain cosystolic expansion for the (co-dimension 1)-skeletons of the KMS complexes.

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1 Introduction

High-dimensional expanders (HDX's) are high-dimensional generalizations of expander graphs. In recent years there has been much work on this subject that has led to several important results, for example, in the analysis of random walks (e.g., [KO20, DDFH24, ALOGV24]), in metric geometry (e.g., [EK24]), in error correcting codes (e.g., [DEL$^+$22, KO22]) and in PCP's (e.g., [BMV24]).

In graphs, expansion is usually defined either via a spectral condition on the eigenvalues of the random walk or as a bound on the Cheeger constant. In the case of graphs, these two definitions are connected (and imply each other) via the Cheeger inequality. In higher dimensions, i.e., considering simplicial complexes in lieu of graphs, there is a spectral definition of expansion (i.e., local spectral expansion – see exact definition in Section 2.2 below) and higher dimensional analogues of the Cheeger constant (i.e., coboundary/cosystolic expansion – see exact definition in Section 2.3 below), but these definitions do not imply one another.

In our work below, we construct new examples of bounded degree cosystolic HDX's. Our construction relies on KMS complexes (which are coset complexes introduced in [GdPVB24]) and on a novel idea of constructing a cone function in an iterative process.

1.1 KMS complexes

In [KO18, KO23], Kaufman and Oppenheim gave a construction of spectral HDX's based on the idea of coset complexes (see below). The construction of Kaufman and Oppenheim was based on the group $\mathrm{SL}_{n+1}(\mathbb{F}_p[t])$, and was later generalized by O'Donnell and Pratt [OP22] to other Chevalley groups.

In [GdPVB24], another coset complex construction of spectral HDX's was given using Kac–Moody–Steinberg groups. Namely, in [GdPVB24] it was shown that given a generalized Cartan matrix satisfying some conditions and a finite field $\mathbb{F}_q$, there is an infinite family of finite coset complexes that are spectral HDX's. The construction of KMS complexes involves some technicalities and we refer the reader to Section 4 for the full details and an explicit example. Our results are restricted to a certain class of KMS complexes characterized as follows. We call a generalized Cartan matrix $A = (A_{ij})_{i,j \in I}$ of rank m n-classical (for $n \leq m$) if every submatrix $(A_{ij})_{i,j \in J}, J \subset I, |J| \leq n$ is of classical (i.e. type A_k, B_k, C_k, D_k) type or if it is reducible, all irreducible parts are of classical type. A KMS complex is called n-classical if its underlying generalized Cartan matrix is n-classical.

While KMS complexes can be seen as a variant of the Chevalley group constructions of [KO23, OP22], they seem to have a significant advantage that is relevant to our result below. Namely, the links of vertices of KMS complexes are associated to opposition complexes which are well-studied simplicial complexes. Opposition complexes are large subcomplexes of spherical buildings and were studied independently to answer questions of finiteness properties of arithmetic groups (see for instance Abramenko [Abr96]). The connection of opposition complexes to KMS complexes is that every link of a KMS complex is either an opposition complex or a join of opposition complexes. Prior work on opposition complexes in [Abr96] suggests that, is some sense, they mimic the properties of spherical buildings. In our work below, this point is explored and we prove that much like spherical buildings, opposition complexes are coboundary expanders.

1.2 Main results

A link of a pure n-dimensional simplicial complex is called *proper* if its dimension is at least 1 and at most $n-1$ (i.e. it is the link of a face of dimension between 0 and $n-2$). As noted above, our main result is that, under some restrictions, proper links of the KMS complexes constructed in [GdPVB24] (see Section 4 below) are coboundary expanders:

Theorem 1.1. *Let $n \geq 3$ be an integer and q be a prime power. There exists $\varepsilon > 0$ such that for every $q > 2^{2n-1}$ and every n-dimensional, n-classical KMS complex X constructed over $k = \mathbb{F}_q$ all the proper links of X are ε-coboundary expanders (with respect to any finitely generated Abelian group as coefficient group).*

The theorem relies on the fact that proper links of n-classical KMS complexes are joins of opposition complexes in spherical buildings of classical type. Given an explicit n-classical KMS group with the extra restriction that no subdiagram of type D_k appears in the Dynkin diagram, it is possible to track ε and get an explicit (but very rough) bound. Tracking an explicit bound for ε in the D_n case could probably be done by very carefully analysing the proof, but we did not attempt to do so.

In [GdPVB24], it was shown that n-dimensional KMS complexes over $\mathbb{F}_q$ are $\frac{2}{\sqrt{q}}$-local spectral expanders for every q that is large enough with respect to n (the KMS complexes are constructed under some assumptions on the corresponding generalized Cartan matrices, see exact formulation in Theorem 4.12 below). Combining this fact with the local-to-global results of [EK24, KM21, DD24] yields the following theorem.

Theorem 1.2. *For an integer $n \geq 3$, let $\{X_i\}_i$ be a family of n-dimensional, n-classical KMS complexes over $\mathbb{F}_q$ (see details in Section 4) and let $\{Y_i\}_i$ be the family of $(n-1)$-dimensional skeletons of $\{X_i\}$. There are $\varepsilon > 0, \mu > 0, q_0 \in \mathbb{N}$ such that if $q \geq q_0$ every Y_i is an (ε, μ)-cosystolic expander (with respect to any finitely generated Abelian group).*

By [DKW18], it follows that the family of complexes $\{Y_i\}_i$ defined in the Theorem above has the topological overlapping property (see exact formulation in [DKW18]).

1.3 Our contributions

There are only few examples of cosystolic HDX's (with a uniformly bounded degree). Indeed, prior to our work, the known constructions were:

- Constructions stemming from quotients of affine buildings (e.g., Ramanujan complexes [LSV05]). These constructions give cosystolic HDX's of every dimension with respect to any finitely generated Abelian coefficient group via the work of Evra and Kaufman [EK24] and the generalizations of Kaufman and Mass [KM21] and of Dikstein and Dinur [DD24]. However, all known such constructions are less symmetric and less elementary than the construction arising from coset complexes (see below).

- Coset complexes arising from Chevalley groups. There are two constructions of this flavour and both only yield 2-dimensional cosystolic HDX's. First, Kaufman and Oppenheim [KO21] showed that their coset complex construction of spectral HDX's gives rise to 2-dimensional cosystolic HDX's with respect to any group coefficients. Second, in a more recent work, O'Donnell and Singer [OS24] showed that coset complexes arising from B_3 Chevalley groups gives rise to 2-dimensional cosystolic HDX's with respect to $\mathbb{F}_2$ coefficients.

Our construction is the first construction of cosystolic HDX's arising from coset complexes that holds beyond dimension 2. Furthermore, our results are more general than previous results on coset complexes arising from Chevalley groups, since our results hold for finitely generated Abelian coefficient group and for KMS complexes of the types $\widetilde{A}_n, \widetilde{B}_n, \widetilde{C}_n, \widetilde{D}_n$.

The method of our proof (as described below) is also novel – we introduce a new method of constructing a cone function by several steps of adding sets of vertices to a subcomplex until we added all the vertices of the entire complex.

1.4 Proof idea

As noted above, Theorem 1.2 follows from Theorem 1.1 via the results of [EK24, KM21, DD24]. Thus, we will focus on the idea of proof of Theorem 1.1, i.e., on the idea of proving that links of KMS complexes are coboundary expanders. Since these links are symmetric, it is enough to show that they have a cone function with bounded radius.

In the constructions of cosystolic expanders stemming from quotients of affine buildings, a cone function was constructed in [LMM16] via utilizing the fact that the links were spherical buildings that have a rich apartment structure. In our construction, the links of KMS complexes do not have such an apartment structure and thus we develop a new method for constructing a cone function with bounded radius. Our method is based on the idea of constructing a cone function by starting with a subcomplex with a cone function, iteratively adding sets of vertices and extending the cone function to that set.

The exact formulation of the theorem is quite technical (see Theorem 3.2 below), and we will explain the idea only on the level of vanishing of homology (a cone function can be thought of as a quantification of vanishing of homology with respect to a base point). Let X be an n-dimensional simplicial complex with a full subcomplex X'. Given a vertex $w \in X$, we denote $X' \cup \{w\}$ to be the full subcomplex of X spanned by the vertices of X' and w. The link of w relative to X', is the intersection $X_w \cap X'$, where X_w is the link of w in X. We will show that vanishing of reduced homology for X' and the link of w relative to X' implies vanishing of reduced homology for $X' \cup \{w\}$. Indeed, let $A = X', B = \{w\} * (X_w \cap X')$ (more precisely the geometric realization of the two subcomplexes, but we will not consider this subtlety at this point). Then

$$X' \cup \{w\} = X' \cup \{\sigma \in X : w \in \sigma, \sigma \setminus \{w\} \in X'\} = A \cup B,$$

and $A \cap B = X_w \cap X'$ (note that the intersection of A and B is in the topological sense, which can be seen as $A \cap B = \{\tau \cap \sigma \mid \tau \in A, \sigma \in B\}$). Furthermore, B can be contracted onto $\{w\}$. Hence we have $\widetilde{H}_i(A) = \widetilde{H}_i(B) = 0$ for $0 \leq i \leq n-1$ and $\widetilde{H}_i(A \cap B) = 0$ for $0 \leq i \leq n-2$. Thus, the Mayer-Vietoris Theorem implies that $\widetilde{H}_i(X) = 0$ for $0 \leq i \leq n-1$.

An important observation is that in the procedure above, instead of adding a single vertex w to X', one can add a set of vertices W to X' as long as no two vertices in W are connected by an edge in X and for every vertex $w \in W$, the link of w relative to X' has vanishing of reduced homology. Theorem 3.2 is a quantification of this idea in the language of cone functions, where the cone radius is bounded as a function of the cone radius of X' and the cone radii of the links of $w \in W$ relative to X'.

Applying this idea to links of KMS complexes follows the work of Abramenko [Abr96] on opposition complexes of spherical buildings of classical type. Let X be an n-dimensional, n-classical KMS complex constructed over $\mathbb{F}_q$. Under this assumption, the links of vertices in X are subcomplexes of classical spherical buildings known as opposition complexes, or joins of such. Roughly speaking, Abramenko showed that if q is sufficiently large, then the links of X can be constructed by the procedure of adding sets of vertices described above, that the number of steps in this procedure is independent of q and that the (relative) links of the added vertices are joins of opposition complexes of spherical buildings of lower dimension. This gives rise to an inductive bound on the cone radii and thus (by the fact that the links are symmetric) on the coboundary expansion of opposition complexes in classical spherical buildings (and thus of the proper links in n-classical KMS complexes).

1.5 Organization

The paper is organized as follows. In Section 2, we recall the notions of local spectral, coboundary and cosystolic high-dimensional expansion, explain the relevant (co)homological background, define cone functions and show some basic properties. This includes the fact that a cone function with coefficients over $\mathbb{Z}$ gives rise to cone functions with coefficients in arbitrary finitely generated Abelian groups, and how one can construct a cone function for the join of two simplicial complexes, given cone functions for each of them separately. In Section 3, we describe our technique of extending cone functions which is a key tool in our work. In Section 4, we recall the construction of the KMS complexes, and in Section 5 we first describe the connection of KMS complexes to buildings and then explain a geometric way to construct classical buildings. Finally, in Section 6 we combine the previous observations to prove Theorem 1.1.

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2 Preliminaries

2.1 Weighted simplicial complexes

Let X be a finite n-dimensional simplicial complex. A simplicial complex X is called *pure n-dimensional* if every face in X is contained in some face of dimension n. The set of all k-faces of X is denoted $X(k)$, and we will be using the convention in which $X(-1) = \{\emptyset\}$. For every $0 \leq k \leq n$, the *k-skeleton of X* is the simplicial complex $\bigcup_{i=-1}^{k} X(i)$. We will say that X is *connected* if its 1-skeleton is a connected graph.

Given a pure n-dimensional simplicial complex X, the *weight function* $w : \bigcup_{k=-1}^{n} X(k) \rightarrow \mathbb{R}_+$ is defined to be

$$\forall \tau \in X(k), w(\tau) = \frac{|\{\sigma \in X(n) : \tau \subseteq \sigma\}|}{\binom{n+1}{k+1} |X(n)|}.$$

We note that w is normalized such that for each $-1 \leq k \leq n$, the function w can be thought of as a probability function on $X(k)$.

Given a finite pure n-dimensional complex X and $\tau \in X$, *the link of* τ is the subcomplex X_τ defined as

$$X_\tau = \{\sigma \in X : \tau \cap \sigma = \emptyset, \tau \cup \sigma \in X\}.$$

To clarify the notation in some situations, we will also write $\text{lk}_X(\tau$ for the link of τ in X. We note that if $\tau \in X(k)$, then X_τ is a pure $(n-k-1)$-dimensional complex. We call X_τ a proper link, if $1 \leq \dim(X_\tau) \leq n-1$.

We denote the weight function w_τ to be the weight function on X_τ defined as above. By slight abuse of notation, we will write $v \in X(0)$ to state that v is a vertex of X.

2.2 Local spectral expansion

Let X be a finite pure n-dimensional simplicial complex with $n \geq 1$ and w the weight function on X defined above. We define the stochastic matrix of the random walk on (the 1-skeleton of) X to be the matrix indexed by $X(0) \times X(0)$ and defined as

$$M(\{v\},\{u\}) = \begin{cases} \frac{w(\{u,v\})}{\sum_{\{u',v\} \in X(1)} w(\{u',v\})} & \{v,u\} \in X(1) \\ 0 & \{v,u\} \notin X(1) \end{cases}.$$

For $\lambda < 1$, we say that X is a *(one-sided) λ-spectral expander* if X is connected and the second largest eigenvalue of M is $\leq \lambda$. We say that X is a *(one-sided) λ-local spectral expander* if for every $-1 \leq k \leq n-2$ and every $\tau \in X(k)$, the simplicial complex X_τ is a (one-sided) λ-spectral expander. We note that every (one-sided) λ-local spectral expander is a (one-sided) λ-spectral expander, since the link of the empty set is X itself.

2.3 Cohomological notations and cosystolic/coboundary expansion

Let X be a pure n-dimensional simplicial complex and $\mathbb{A}$ an Abelian group.

We denote $\overrightarrow{X}(k)$ to be the set of oriented k-simplices and for $\sigma \in \overrightarrow{X}(k)$, we denote $-\sigma$ to be the simplex with reversed orientation from σ on the same vertex set. For $\{v_0,\ldots,v_k\} \in X(k)$, we denote $[v_0,\ldots,v_k] \in \overrightarrow{X}(k)$ to be the oriented simplex with the orientation induced from the ordering of the vertices. Note that for every permutation $\gamma \in \text{Sym}\{0,\ldots,k\}$ it holds that

$$[v_{\gamma(0)},\ldots,v_{\gamma(k)}] = (-1)^{\text{sign}(\gamma)}[v_0,\ldots,v_k].$$

We denote $C^k(X;\mathbb{A})$ to be the set of k-cochains defined as follows: $C^k(X;\mathbb{A})$ is the set of functions $\phi : \overrightarrow{X}(k) \rightarrow \mathbb{A}$ that are anti-symmetric, i.e., for every $[v_0,\ldots,v_k]$, $\phi(-[v_0,\ldots,v_k]) = -\phi([v_0,\ldots,v_k])$. We recall that $C^k(X;\mathbb{A})$ is an Abelian group with respect to pointwise addition.

Let w be the weight function on X defined above. For $\phi \in C^k(X;\mathbb{A})$, we define

$$\text{supp}(\phi) = \{\{v_0,\ldots,v_k\} \in X(k) : \phi([v_0,\ldots,v_k]) \neq 0_{\mathbb{A}}\},$$
$$\|\phi\| = \sum_{\sigma \in \text{supp}(\phi)} w(\sigma).$$

Given a subgroup $K \subseteq C^k(X;\mathbb{A})$ and a cochain $\phi \in C^k(X;\mathbb{A})$, we define

$$\|\phi - K\| = \min_{\psi \in K} \|\phi - \psi\|.$$

The *coboundary map* $d_k : C^k(X;\mathbb{A}) \rightarrow C^{k+1}(X;\mathbb{A})$ is the map defined by mapping $\phi \in C^k(X;\mathbb{A})$ to $d_k\phi$ given by

$$(d_k\phi)([v_0,\ldots,v_{k+1}]) = \sum_{i=0}^{k+1} (-1)^i \phi([v_0,\ldots,\hat{v_i},\ldots,v_{k+1}]),$$

where $[v_0, \ldots, \hat{v}_i, \ldots, v_{k+1}]$ is the simplex obtained from $[v_0, \ldots, v_{k+1}]$ by removing the vertex v_i. We denote $B^k(X; \mathbb{A}) = \mathrm{Im}(d_{k-1})$ and $Z^k(X; \mathbb{A}) = \mathrm{Ker}(d_k)$ and recall that these are Abelian subgroups of $C^k(X; \mathbb{A})$ and that $B^k(X; \mathbb{A}) \subseteq Z^k(X; \mathbb{A})$. The k-cohomology $H^k(X; \mathbb{A})$ is defined as $H^k(X; \mathbb{A}) = Z^k(X; \mathbb{A})/B^k(X; \mathbb{A})$.

The k-coboundary/cosystolic expansion constant of X (with coefficients in $\mathbb{A}$) are defined to be

$$h^k_{\mathrm{cb}}(X; \mathbb{A}) = \min_{\phi \in C^k(X;\mathbb{A}) \setminus B^k(X;\mathbb{A})} \frac{\|d_k \phi\|}{\|\phi - B^k(X; \mathbb{A})\|},$$

$$h^k_{\mathrm{cs}}(X; \mathbb{A}) = \min_{\phi \in C^k(X;\mathbb{A}) \setminus Z^k(X;\mathbb{A})} \frac{\|d_k \phi\|}{\|\phi - Z^k(X; \mathbb{A})\|}.$$

We note that be definition $h^k_{\mathrm{cs}}(X; \mathbb{A}) \geq h^k_{\mathrm{cb}}(X; \mathbb{A})$ and that by definition $h^k_{\mathrm{cs}}(X; \mathbb{A}) > 0$. We also note that $h^k_{\mathrm{cs}}(X; \mathbb{A}) = h^k_{\mathrm{cb}}(X; \mathbb{A})$ if and only if $H^k(X; \mathbb{A}) = 0$ and that if $H^k(X; \mathbb{A}) \neq 0$, then $h^k_{\mathrm{cb}}(X; \mathbb{A}) = 0$.

For a finite pure n-dimensional simplicial complex X and a constant $\varepsilon > 0$, we will say that X is a $(\varepsilon, \mathbb{A})$*-coboundary expander* if for every $0 \leq k \leq n-1$, $h^k_{\mathrm{cb}}(X; \mathbb{A}) \geq \varepsilon$. Also, for constants $\varepsilon > 0, \mu > 0$, we will say that X *is a* $(\varepsilon, \mu, \mathbb{A})$*-cosystolic expander* if for every $0 \leq k \leq n-1$, $h^k_{\mathrm{cs}}(X; \mathbb{A}) \geq \varepsilon$ and for every $\phi \in Z^k(X, \mathbb{A}) \setminus B^k(X, \mathbb{A})$, $\|\phi\| \geq \mu$.

In [KM21, DD24] the following local to global results were proven:

Theorem 2.1. *Let $n \geq 3$ be an integer, $\varepsilon' > 0$ a constant and $\mathbb{A}$ be a finitely generated Abelian group. There are constants $0 < \lambda < 1, \varepsilon > 0, \mu > 0$ such that the following holds: For every finite pure n-dimensional simplicial complex X if*

- *the simplicial complex X is a (one-sided) λ-local spectral expander,*

and

- *for every $0 \leq k \leq n-2$ and every $\tau \in X(k)$, it holds that X_τ is a $(\varepsilon', \mathbb{A})$-coboundary expander,*

then the $(n-1)$-skeleton of X is a $(\varepsilon, \mu, \mathbb{A})$-cosystolic expander.

2.4 Homological notations and cone functions

Let X be a pure n-dimensional simplicial complex and $\mathbb{A}$ an Abelian group.

Keeping the notations regrading oriented simplices above, we denote $C_k(X; \mathbb{A})$ to be the set of k-chains defined as follows: $C_k(X; \mathbb{A})$ is the set of all formal sums of the form

$$\sum_{\sigma \in \overrightarrow{X}(k)} t_\sigma \sigma,$$

such that $t_\sigma \in \mathbb{A}$ and $t_{-\sigma} = -t_\sigma$. We note that $C_k(X; \mathbb{A})$ is an Abelian group or equivalently a $\mathbb{Z}$-module. For $A \in C_k(X; \mathbb{A})$, we define

$$\mathrm{supp}(A) = \{\{v_0, \ldots, v_k\} : t_{[v_0, \ldots, v_k]} \neq 0_{\mathbb{A}}\}.$$

In the sequel, it will be useful to work with a "basis" for $C_k(X; \mathbb{A})$. For any $\sigma_0 \in \overrightarrow{X}(k)$ and any $a \in \mathbb{A}$, we define $a\mathbb{1}_{\sigma_0} \in C_k(X; \mathbb{Z})$ to be

$$a\mathbb{1}_\sigma = \sum_{\sigma \in \overrightarrow{X}(k)} t_\sigma \sigma$$

with

$$t_\sigma = \begin{cases} a & \sigma = \sigma_0 \\ -a & \sigma = -\sigma_0 \\ 0_{\mathbb{A}} & \text{otherwise} \end{cases}.$$

With this notation, every element in $C_k(X;\mathbb{A})$ is a formal sum of elements of the form $a\mathbb{1}_\sigma, \sigma \in \overrightarrow{X}(k), a \in \mathbb{A}$.

The *boundary map* $\partial_k : C_k(X;\mathbb{A}) \to C_{k-1}(X;\mathbb{A})$ is the map given by

$$\partial_k \left(\sum_{[v_0,\ldots,v_k] \in \overrightarrow{X}(k)} t_{[v_0,\ldots,v_k]} \mathbb{1}_{[v_0,\ldots,v_k]} \right) = \sum_{[v_0,\ldots,v_k] \in \overrightarrow{X}(k)} \sum_{i=0}^{k} (-1)^i t_{[v_0,\ldots,v_k]} \mathbb{1}_{[v_0,\ldots,\hat{v}_i,\ldots,v_k]}.$$

Definition 2.2 (Cone function)**.** *Let X be a finite simplicial complex. Let $k \in \mathbb{N} \cup \{0,-1\}$ be a constant, $\mathbb{A}$ an Abelian group and v be a vertex of X. A $(k,\mathbb{A})$-cone function with apex v is a function* $\mathrm{Cone}_X : \bigoplus_{j=-1}^{k} C_j(X;\mathbb{A}) \to \bigoplus_{j=0}^{k+1} C_j(X;\mathbb{A})$ *that fulfills the following conditions:*

1. *For every $a \in \mathbb{A}$,* $\mathrm{Cone}_X(a\mathbb{1}_\emptyset) = a\mathbb{1}_{[v]}$.
2. *For every $-1 \leq j \leq k$,* $\mathrm{Cone}_X(C_j(X;\mathbb{A})) \subseteq C_{j+1}(X;\mathbb{A})$ *and*

$$\mathrm{Cone}_X|_{C_j(X;\mathbb{A})} : C_j(X;\mathbb{A}) \to C_{j+1}(X;\mathbb{A})$$

   is a homomorphism of $\mathbb{Z}$-modules.
3. *For every $0 \leq j \leq k$ and every $A \in C_j(X;\mathbb{A})$,*

$$\partial_{j+1} \mathrm{Cone}_X(A) + \mathrm{Cone}_X(\partial_j A) = A.$$

Below, we will refer to

$$\partial_{j+1} \mathrm{Cone}_X(A) + \mathrm{Cone}_X(\partial_j A) = A$$

as *the cone equation.*

Definition 2.3 (Radius of a cone)**.** *Let X be an n-dimensional simplicial complex and $-1 \leq k \leq n-1$. Given a $(k,\mathbb{A})$-cone function* Cone_X*, we define for every $-1 \leq j \leq k$, the j-th radius of* Cone_X *to be*

$$\mathrm{Rad}_j(\mathrm{Cone}_X) = \max_{\sigma \in \overrightarrow{X}(j), a \in \mathbb{A}} |\mathrm{supp}(\mathrm{Cone}_X(a\mathbb{1}_\sigma))|.$$

Define the $(k,\mathbb{A})$-th cone radius of X to be

$$\mathrm{Rad}_k(X,\mathbb{A}) = \min\{\mathrm{Rad}_k(\mathrm{Cone}_X) : \mathrm{Cone}_X \text{ is a } (k,\mathbb{A})\text{-cone function with } \}.$$

If no $(k,\mathbb{A})$-cone function of X exists, we define $\mathrm{Rad}_k(X,\mathbb{A}) = \infty$.

Remark 2.4. *Note that in the case where $n = 0$ and X is a just a non-empty set of vertices, it holds by definition that* $\mathrm{Rad}_{-1}(X,\mathbb{A}) = 1$.

Example 2.5. Let Γ be a simplicial complex of dimension 1, i.e. a graph. Assume that Γ is connected. Then we can construct a $(0, \mathbb{A})$-cone function for Γ in the following way. Pick an arbitrary vertex $v \in \Gamma(0)$ and set $\mathrm{Cone}_\gamma(\mathbb{1}_\emptyset) = \mathbb{1}_{[v]}$. Furthermore, we define $\mathrm{Cone}_\Gamma(\mathbb{1}_{[v]}) = 0$. In this case we have

$$\partial\, \mathrm{Cone}_\Gamma(\mathbb{1}_{[v]}) = 0 = \mathbb{1}_{[v]} - \mathrm{Cone}_\Gamma(\mathbb{1}_\emptyset) = \mathbb{1}_{[v]} - \mathrm{Cone}_\Gamma(\partial \mathbb{1}_{[v]})$$

and the cone equation is satisfied in this case.

Next, let $w \in \Gamma(0), w \neq v$. Since Γ is connected, there exists a path $w = w_0, \ldots, w_n = v$ from w to v. We define

$$\mathrm{Cone}_\Gamma(\mathbb{1}_{[w]}) = \sum_{i=1}^{n} \mathbb{1}_{[w_{i-1}, w_i]}.$$

Then

$$\partial\, \mathrm{Cone}_\Gamma(\mathbb{1}_{[w]}) = \sum_{i=1}^{n} \mathbb{1}_{[w_{i-1}]} - \mathbb{1}_{[w_i]} = \mathbb{1}_{[w_0]} - \mathbb{1}_{[w_n]} = \mathbb{1}_{[w]} - \mathbb{1}_{[v]} = \mathbb{1}_{[w]} - \mathrm{Cone}_\Gamma(\mathbb{1}_\emptyset) = \mathbb{1}_{[w]} - \mathrm{Cone}_\Gamma(\partial \mathbb{1}_{[w]})$$

which shows that the cone equation is satisfied in all cases, since we defined Cone_Γ on the generators of $C_0(\Gamma, A)$ and extend it to a homomorphism.

Additionally, note that if we choose the path from w to v to be of minimal length we have

$$\mathrm{Rad}_{-1}(\mathrm{Cone}_\Gamma) = 1, \qquad \mathrm{Rad}_0(\mathrm{Cone}_\Gamma) \leq \max_{w \in \Gamma(0)} \mathrm{dist}(v, w)$$

where $\mathrm{dist}(v, w)$ is the minimum length of a path between v, w.

We will show that a bound on the cone radii of $\mathbb{Z}$-cones bounds the cone radii for every finitely generated Abelian group. This result is probably well known to experts, but we could not find a reference for it, so we include a sketch of the proof here (leaving some of the details for the reader).

Proposition 2.6. *Let X be an n-dimensional simplicial complex and $-1 \leq k \leq n-1$. For every finitely generated Abelian group $\mathbb{A}$, it holds that $\mathrm{Rad}_k(X, \mathbb{A}) \leq \mathrm{Rad}_k(X, \mathbb{Z})$.*

Proof. If $(k, \mathbb{Z})$-cone functions of X do not exist, then $\mathrm{Rad}_k(X, \mathbb{Z}) = \infty$ and the inequality holds trivially. Assume that there exists a $(k, \mathbb{Z})$-cone function.

By the fundamental Theorem of finitely generated Abelian groups, there are prime powers $q_1, \ldots, q_l$ and an integer $m \in \mathbb{N} \cup \{0\}$ such that $\mathbb{A} = \left(\bigoplus_{j=1}^{l} \mathbb{Z}/(q_j\mathbb{Z})\right) \oplus \mathbb{Z}^m$. We will denote the elements of $\mathbb{A}$ as $m + l$ tuples $(a_1, \ldots, a_l, a_{l+1}, \ldots, a_{l+m})$, where for every $1 \leq j \leq l$, $a_j \in \mathbb{Z}/(q_j\mathbb{Z})$ and $a_{l+1}, \ldots, a_{l+m} \in \mathbb{Z}$. We note that for every $-1 \leq i \leq n$,

$$C_i(X, \mathbb{A}) = \left(\bigoplus_{j=1}^{l} C_i(X, \mathbb{Z}/(q_j\mathbb{Z}))\right) \oplus C_i(X, \mathbb{Z})^m.$$

Explicitly, for every $\sigma \in \overrightarrow{X}(i)$ and every $(a_1, \ldots, a_{l+m}) \in \mathbb{A}$, we identify $(a_1, \ldots, a_{l+m})\mathbb{1}_\sigma \in C_i(X, \mathbb{A})$ with $\bigoplus_{j=1}^{l+m} a_j \mathbb{1}_\sigma \in \left(\bigoplus_{j=1}^{l} C_i(X, \mathbb{Z}/(q_j\mathbb{Z}))\right) \oplus C_i(X, \mathbb{Z})^m$.

For $1 \leq j \leq l$, let $\varphi_j : \mathbb{Z} \to \mathbb{Z}/(q_j\mathbb{Z})$ be the quotient map. For every $-1 \leq i \leq n$, φ_j extends to a surjective homomorphism $\varphi_j : C_i(X, \mathbb{Z}) \to C_i(X, \mathbb{Z}/(q_j\mathbb{Z}))$ (this is a homomorphism of $\mathbb{Z}$-modules).

Let Cone_X be a $(k,\mathbb{Z})$-cone function. For every $1 \leq j \leq l$, we define the function

$$\text{Cone}_X^j : \bigoplus_{i=-1}^{k} C_i(X;\mathbb{Z}/(q_j\mathbb{Z})) \rightarrow \bigoplus_{i=0}^{k+1} C_i(X;\mathbb{Z}/(q_j\mathbb{Z}))$$

as follows:

$$\text{Cone}_X^j(\varphi_j(A)) = \varphi_j(\text{Cone}_X(A)), \forall -1 \leq i \leq k, \forall A \in C_i(X;\mathbb{Z}).$$

We note that Cone_X^j is well-defined: Let $A, A' \in C_i(X;\mathbb{Z})$ with $\varphi_j(A) = \varphi_j(A')$. Then $A - A' \in \text{Ker}(\varphi_j)$, which implies that there is $A'' \in C_i(X;\mathbb{Z})$ such that $A - A' = q_j A''$. Thus

$$\text{Cone}_X^j(\varphi_j(A)) - \text{Cone}_X^j(\varphi_j(A')) = \varphi_j(\text{Cone}_X(A)) - \varphi_j(\text{Cone}_X(A')) =$$
$$\varphi_j(\text{Cone}_X(A - A')) = \varphi_j(\text{Cone}_X^{\mathbb{Z}}(q_j A'')) = \varphi_j(q_j \,\text{Cone}_X^{\mathbb{Z}}(A'')) = 0_{\mathbb{Z}/(q_j\mathbb{Z})}$$

and it follows that $\text{Cone}_X^j(\varphi_j(A)) = \text{Cone}_X^j(\varphi_j(A'))$. Also, by the fact that φ_j is surjective on every $C_i(X,\mathbb{Z}/(q_j\mathbb{Z}))$ it follows that Cone_X^j is defined on every $A \in C_i(X;\mathbb{Z}/(q_j\mathbb{Z}))$.

We leave it to the reader to verify that Cone_X^j is a $(k,\mathbb{Z}/(q_j\mathbb{Z}))$-cone function and note that for every i, every $\sigma \in \overrightarrow{X}(i)$ and every $a \in \mathbb{Z}/(q_j\mathbb{Z})$ it holds that

$$\text{supp}(\text{Cone}_X^j(a\mathbb{1}_\sigma)) \subseteq \text{supp}(\text{Cone}_X(\mathbb{1}_\sigma)).$$

For every $l+1 \leq j \leq m+l$, we denote $\text{Cone}_X^j = \text{Cone}_X$ (this is a $(k,\mathbb{Z})$-cone function for every $l+1 \leq j \leq l+m$). For every $-1 \leq i \leq k$ and every

$$\bigoplus_{j=1}^{l+m} A_j \in \left(\bigoplus_{j=1}^{l} C_i(X,\mathbb{Z}/(q_j\mathbb{Z})) \right) \oplus C_i(X,\mathbb{Z})^m,$$

we define

$$\text{Cone}_X^{\mathbb{A}}(\bigoplus_{j=1}^{l+m} A_j) = \bigoplus_{j=1}^{l+m} \text{Cone}_X^j(A_j).$$

Extending this map $\mathbb{Z}$-linearly yields a map

$$\text{Cone}_X^{\mathbb{A}} : \bigoplus_{i=-1}^{k} \left(\left(\bigoplus_{j=1}^{l} C_i(X,\mathbb{Z}/(q_j\mathbb{Z})) \right) \oplus C_i(X,\mathbb{Z})^m \right) \rightarrow \bigoplus_{i=0}^{k+1} \left(\left(\bigoplus_{j=1}^{l} C_i(X,\mathbb{Z}/(q_j\mathbb{Z})) \right) \oplus C_i(X,\mathbb{Z})^m \right),$$

i.e., a map

$$\text{Cone}_X^{\mathbb{A}} : \bigoplus_{i=-1}^{k} C_i(X,\mathbb{A}) \rightarrow \bigoplus_{i=0}^{k+1} C_i(X,\mathbb{A}).$$

We conclude the proof by observing that $\text{Cone}_X^{\mathbb{A}}$ is a $(k,\mathbb{A})$-cone function that that for every $(a_1,\ldots,a_{l+m}) \in \mathbb{A}$, every $-1 \leq i \leq k$ and every $\sigma \in \overrightarrow{X}(i)$, it holds that

$$\text{supp}(\text{Cone}_X^{\mathbb{A}}((a_1,\ldots,a_{l+m})\mathbb{1}_\sigma) \subseteq \text{supp}(\text{Cone}_X(\mathbb{1}_\sigma)).$$

Since Cone_X was an arbitrary $(k,\mathbb{Z})$-cone function, it follows that $\text{Rad}_k(X,\mathbb{A}) \leq \text{Rad}_k(X,\mathbb{Z})$.

□

In light of this Proposition, in the sequel, we will be interested with bounding $\mathbb{Z}$-radii for cone functions. Below, we will omit the $\mathbb{Z}$ notation when discussing cone functions: we will write k-cone functions in lieu of $(k, \mathbb{Z})$-cone functions, $\mathrm{Rad}_k(X)$ in lieu of $\mathrm{Rad}_k(X, \mathbb{Z})$ etc.

Observation 2.7. *We observe that for every j, $C_j(X; \mathbb{Z})$ is a free $\mathbb{Z}$-module and thus the condition in the definition of the cone function that*

$$\mathrm{Cone}_X|_{C_j(X;\mathbb{Z})} : C_j(X; \mathbb{Z}) \to C_{j+1}(X; \mathbb{Z})$$

is a homomorphism is equivalent to the condition that this map is a $\mathbb{Z}$-linear map of $\mathbb{Z}$-modules.

By this linearity, the cone equation is equivalent to the condition:

$$\partial_{j+1} \mathrm{Cone}_X(\mathbb{1}_\sigma) + \mathrm{Cone}_X(\partial_j \mathbb{1}_\sigma) = \mathbb{1}_\sigma, \forall \sigma \in \overrightarrow{X}(j).$$

In other words, in order to verify that a linear function is a cone function, it is enough to check the cone equation on $\mathbb{1}_\sigma$ for every oriented simplex σ.

Moreover, when one defines a cone function, it is enough to define it as a linear function that fulfills the cone equation on $\mathbb{1}_\sigma$ on the subset of oriented simplices that contain at least one orientation for every simplex.

The work of Gromov [Gro10] and subsequent works [KO21, KM19] showed that for symmetric simplicial complexes, a bound on the cone radius gives rise to a bound on the coboundary expansion. For a simplicial complex X, we denote $\mathrm{Aut}(X)$ to be the simplicial automorphisms from X to itself. In [KO21] the following was proven:

Theorem 2.8. *Let $n \geq 1$, $\mathcal{R} > 0$ be constants and $\mathbb{A}$ an Abelian group. For every finite pure n-dimensional simplicial complex, if*

- *the group $\mathrm{Aut}(X)$ acts transitively on $X(n)$,*

and

- *for every $0 \leq k \leq n-1$ it holds that $\mathrm{Rad}_k(X, \mathbb{A}) \leq \mathcal{R}$,*

then for every $0 \leq k \leq n-1$, $h^k_{\mathrm{cb}}(X, \mathbb{A}) \geq \frac{1}{\mathcal{R}\binom{n+1}{k+1}}$.

Combining this Theorem with Proposition 2.6 yields the following:

Theorem 2.9. *Let $n \geq 1$, $\mathcal{R} > 0$ be constants. For every finite pure n-dimensional simplicial complex, if*

- *The group $\mathrm{Aut}(X)$ acts transitively on $X(n)$*

and

- *For every $0 \leq k \leq n-1$ it holds that $\mathrm{Rad}_k(X, \mathbb{Z}) \leq \mathcal{R}$*

Then for every $0 \leq k \leq n-1$ and every finitely generate Abelian group $\mathbb{A}$ it holds that

$$h^k_{\mathrm{cb}}(X, \mathbb{A}) \geq \frac{1}{\mathcal{R}\binom{n+1}{k+1}}.$$

Remark 2.10. *The statement in [KO21] is for $\mathbb{A} = \mathbb{F}_2$, but the proof given there generalizes almost verbatim for a general Abelian group.*

2.5 Cone radius of joins

In this section, we denote $\sqcup$ to be a disjoint union. Given two non-empty finite simplicial complexes Y_1 and Y_2 with vertex sets $V(Y_1)$ and $V(Y_2)$, the *join* of Y_1 and Y_2, denoted $Y_1 * Y_2$, is the simplicial complex with the vertex set $V(Y_1) \sqcup V(Y_2)$ and simplices $\sigma \sqcup \tau$ for every $\sigma \in Y_1$ and $\tau \in Y_2$. We note that $Y_1 * Y_2$ includes all the simplices of the form $\tau = \emptyset \sqcup \tau$ and $\sigma = \sigma \sqcup \emptyset$ for every $\sigma \in Y_1$ and $\tau \in Y_2$.

Below, we will need the following definition (which will also be used in the next section): Given a simplicial complex X and a vertex v in X, we define the following operation; For every $0 \leq j \leq k-1$ and every $[v_0, \ldots, v_j] \in \overrightarrow{X_v}(j)$, we define

$$[v, \mathbb{1}_{[v_0,\ldots,v_j]}] = \mathbb{1}_{[v,v_0,\ldots,v_j]}.$$

Extending this definition linearly, for every $A \in C_j(X_v; \mathbb{Z})$, we define $[v, A] \in C_{j+1}(X; \mathbb{Z})$. We observe that for every $[v_0, \ldots, v_j] \in \overrightarrow{X_v}(j)$,

$$\partial_{j+1} \mathbb{1}_{[v,v_0,\ldots,v_j]} = \mathbb{1}_{[v,v_0,\ldots,v_j]} - [v, \partial_j \mathbb{1}_{[v_0,\ldots,v_j]}]$$

and thus by linearity, for every $A \in C_j(X_v; \mathbb{Z})$,

$$\partial_{j+1}[v, A] = A - [v, \partial_j A]. \tag{1}$$

Proposition 2.11. *Let Y be a non-empty finite simplicial complex and v a vertex possibly not in Y. Then for every $k \in \mathbb{N} \cup \{0\}$, there is a k-cone function* $\text{Cone}_{\{v\}*Y}$ *such that for every $-1 \leq j \leq k$,* $\text{Rad}_j(\text{Cone}_{\{v\}*Y}) \leq 1$.

Proof. Fix $k \in \mathbb{N} \cup \{0\}$ and define a function $\text{Cone}_{\{v\}*Y}$ as follows:

- $\text{Cone}_{\{v\}*Y}(\emptyset) = \mathbb{1}_{[v]}$.
- For every $0 \leq j \leq k$ and every $[v_0, \ldots, v_j] \in \overrightarrow{Y}(j)$,
$$\text{Cone}_{\{v\}*Y}(\mathbb{1}_{[v_0,\ldots,v_j]}) = [v, \mathbb{1}_{[v_0,\ldots,v_j]}] = \mathbb{1}_{[v,v_0,\ldots,v_j]}.$$
- For every $0 \leq j \leq k$ and every $\{v_1, \ldots, v_j\} \in Y$,
$$\text{Cone}_{\{v\}*Y}(\mathbb{1}_{[v,v_1,\ldots,v_j]}) = 0.$$

Extend the function $\text{Cone}_{\{v\}*Y}$ linearly.

We check that the cone equation holds for the above function. In the case of $[v_0, \ldots, v_j] \in \overrightarrow{Y}(j)$ this is due to (1). For $\{v_1, \ldots, v_j\} \in Y$, we note that

$$\begin{aligned}
&\partial_{j+1} \text{Cone}_{\{v\}*Y}(\mathbb{1}_{[v,v_1,\ldots,v_j]}) + \text{Cone}_{\{v\}*Y}(\partial_j \mathbb{1}_{[v,v_1,\ldots,v_j]}) = \\
&0 + \text{Cone}_{\{v\}*Y}(\mathbb{1}_{[v_1,\ldots,v_j]}) - \text{Cone}_{\{v\}*Y}([v, \partial_{j-1}[v_1, \ldots, v_j]) = \\
&0 + \mathbb{1}_{[v,v_1,\ldots,v_j]} + 0 = \mathbb{1}_{[v,v_1,\ldots,v_j]}
\end{aligned}$$

as needed. Thus, by Observation 2.7, the above function is a k-cone function and it is easy to see that $\text{Rad}_j(\text{Cone}_{\{v\}*Y}) \leq 1$ for every $-1 \leq j \leq k$. $\square$

Remark 2.12. *We note that in the above construction, the cone function is well-defined for every k, i.e., it is well-defined even in the cases where $k \geq \dim(\{v\} * Y)$.*

Given a simplicial complex X with $\{v_0, \ldots, v_{j_1}\}, \{u_0, \ldots, u_{j_2}\} \in X$ such that $\{v_0, \ldots, v_{j_1}\} \cap \{u_0, \ldots, u_{j_2}\} = \emptyset$ and $\{v_0, \ldots, v_{j_1}\} \cup \{u_0, \ldots, u_{j_2}\} \in X$, we define for $\sigma = [v_0, \ldots, v_{j_1}]$ and $\tau = [u_0, \ldots, u_{j_2}]$,

$$[\sigma, \tau] = [v_0, \ldots, v_{j_1}, u_0, \ldots, u_{j_2}] \in \overrightarrow{X}(j_1 + j_2 + 1)$$

and

$$[\mathbb{1}_\sigma, \mathbb{1}_\tau] = \mathbb{1}_{[\sigma,\tau]}.$$

For $X = Y_1 * Y_2$, we note that for every $\sigma \in \overrightarrow{Y_1}(j_1)$ and $\tau \in \overrightarrow{Y_2}(j_2)$, $[\mathbb{1}_\sigma, \mathbb{1}_\tau]$ is well-defined. Also note that for σ and τ as above,

$$\partial_{j_1+j_2+1}[\mathbb{1}_\sigma, \mathbb{1}_\tau] = [\partial_{j_1}\mathbb{1}_\sigma, \mathbb{1}_\tau] + (-1)^{j_1+1}[\mathbb{1}_\sigma, \partial_{j_2}\mathbb{1}_\tau].$$

Extending this equation linearly on $C_{j_1}(Y_1; \mathbb{Z})$ and $C_{j_2}(Y_2; \mathbb{Z})$ yields that for every $A_1 \in C_{j_1}(Y_1; \mathbb{Z})$ and every $A_2 \in C_{j_2}(Y_2; \mathbb{Z})$,

$$\partial_{j_1+j_2+1}[A_1, A_2] = [\partial_{j_1}A_1, A_2] + (-1)^{j_1+1}[A_1, \partial_{j_2}A_2]. \tag{2}$$

We note that this equation is actually a generalization of (1) above.

Furthermore, observe that for $\text{supp}([A_1, A_2]) = \text{supp}(A_1) \times \text{supp}(A_2)$.

The main idea of following Proposition is taken from P. Wild's PhD Thesis [Wil22] and we claim no originality here.

Proposition 2.13. *Let Y_1, Y_2 be finite simplicial complexes of dimension n_1, n_2 correspondingly. Assume that for $i = 1, 2$ there is an n_i-cone function Cone_{Y_i} and constants $R_j^{(i)} \in \mathbb{N}, -1 \leq j \leq n_i - 1$ such that $\text{Rad}_j(\text{Cone}_{Y_i}) \leq R_j^i$. Then there is an $(n_1 + n_2)$-cone function $\text{Cone}_{Y_1 * Y_2}$ such that for every $-1 \leq j \leq n_1 + n_2$,*

$$\text{Rad}_j(\text{Cone}_{Y_1 * Y_2}) \leq \begin{cases} \max_{-1 \leq j_1 \leq j} R_{j_1}^{(1)} & j < n_1 \\ \max\{\max_{-1 \leq j_1 \leq n_1 - 1} R_{j_1}^{(1)}, ((n_1 + 1)R_{n_1-1}^{(1)} + 1)R_{j-n_1-1}^{(2)}\} & n_1 \leq j \leq n_1 + n_2 \end{cases}.$$

Proof. Let v be the apex of Cone_{Y_1}. Define

$$\text{Cone}_{Y_1 * Y_2}(\mathbb{1}_\emptyset) = \mathbb{1}_{[v]}.$$

Let $-1 \leq j_1 \leq n_1$ and $-1 \leq j_2 \leq n_2$ such that $0 \leq j_1 + j_2 + 1 \leq n_1 + n_2$. For $\sigma_i \in \overrightarrow{X}(j_i), i = 1, 2$, we define

$$\text{Cone}_{Y_1 * Y_2}(\mathbb{1}_{[\sigma_1, \sigma_2]}) = \begin{cases} [\text{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}), \mathbb{1}_{\sigma_2}] & \dim(\sigma_1) < n_1 \\ (-1)^{n_1+1}[\mathbb{1}_{\sigma_1} - \text{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}), \text{Cone}_{Y_2}(\mathbb{1}_{\sigma_2})] & \dim(\sigma_1) = n_1 \end{cases}$$

and extend it linearly.

We will check that the cone equation is fulfilled. We note that by linearity,

$$\begin{aligned}
&\mathrm{Cone}_{Y_1*Y_2}(\partial_{j_1+j_2+1}\mathbb{1}_{[\sigma_1,\sigma_2]}) = \\
&\mathrm{Cone}_{Y_1*Y_2}([\partial_{j_1}\mathbb{1}_{\sigma_1},\mathbb{1}_{\sigma_2}]) + (-1)^{j_1+1}\mathrm{Cone}_{Y_1*Y_2}([\mathbb{1}_{\sigma_1},\partial_{j_2}\mathbb{1}_{\sigma_2}]) = \\
&\begin{cases} [\mathrm{Cone}_{Y_1}(\partial_{j_1}\mathbb{1}_{\sigma_1}),\mathbb{1}_{\sigma_2}] + (-1)^{j_1+1}[\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}),\partial_{j_2}\mathbb{1}_{\sigma_2}] & \dim(\sigma_1) < n_1 \\ [\mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}),\mathbb{1}_{\sigma_2}] + (-1)^{n_1+1}(-1)^{n_1+1}[\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}),\mathrm{Cone}_{Y_2}(\partial_{j_2}\mathbb{1}_{\sigma_2})] & \dim(\sigma_1) = n_1 \end{cases} = \\
&\begin{cases} [\mathrm{Cone}_{Y_1}(\partial_{j_1}\mathbb{1}_{\sigma_1}),\mathbb{1}_{\sigma_2}] + (-1)^{j_1+1}[\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}),\partial_{j_2}\mathbb{1}_{\sigma_2}] & \dim(\sigma_1) < n_1 \\ [\mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}),\mathbb{1}_{\sigma_2}] + [\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}),\mathrm{Cone}_{Y_2}(\partial_{j_2}\mathbb{1}_{\sigma_2})] & \dim(\sigma_1) = n_1 \end{cases}.
\end{aligned}$$

Assume first that $\dim(\sigma_1) < n_1$. Then

$$\begin{aligned}
&\partial_{j_1+j_2+2}\mathrm{Cone}_{Y_1*Y_2}(\mathbb{1}_{[\sigma_1,\sigma_2]}) =^{(2)} \\
&[\partial_{j_1+1}\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}),\mathbb{1}_{\sigma_2}] + (-1)^{j_1+2}[\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}),\partial_{j_2}\mathbb{1}_{\sigma_2}] = \\
&[\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{j_1}\mathbb{1}_{\sigma_1}),\mathbb{1}_{\sigma_2}] + (-1)^{j_1+2}[\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}),\partial_{j_2}\mathbb{1}_{\sigma_2}] = \\
&[\mathbb{1}_{\sigma_1},\mathbb{1}_{\sigma_2}] - \big([\mathrm{Cone}_{Y_1}(\partial_{j_1}\mathbb{1}_{\sigma_1}),\mathbb{1}_{\sigma_2}] + (-1)^{j_1+1}[\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}),\partial_{j_2}\mathbb{1}_{\sigma_2}]\big) = \\
&\mathbb{1}_{[\sigma_1,\sigma_2]} - \mathrm{Cone}_{Y_1*Y_2}(\partial_{j_1+j_2+1}\mathbb{1}_{[\sigma_1,\sigma_2]}).
\end{aligned}$$

Next, assume that $\dim(\sigma_1) = n_1$. Note that

$$\partial_{n_1}\left(\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1})\right) = \partial_{n_1}\mathbb{1}_{\sigma_1} - (\partial_{n_1}\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1-1}\partial_{n_1}\mathbb{1}_{\sigma_1})) = 0. \tag{3}$$

Then

$$\begin{aligned}
&\partial_{n_1+j_2+2}\mathrm{Cone}_{Y_1*Y_2}(\mathbb{1}_{[\sigma_1,\sigma_2]}) =^{(2)} \\
&(-1)^{n_1+1}[\partial_{n_1}\left(\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1})\right),\mathrm{Cone}_{Y_2}(\mathbb{1}_{\sigma_2})] + \\
&(-1)^{n_1+1}(-1)^{n_1+1}[\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}),\partial_{j_2+1}\mathrm{Cone}_{Y_2}(\mathbb{1}_{\sigma_2})] =^{(3)} \\
&[\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}),\partial_{j_2+1}\mathrm{Cone}_{Y_2}(\mathbb{1}_{\sigma_2})] = \\
&[\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}),\mathbb{1}_{\sigma_2} - \mathrm{Cone}_{Y_2}(\partial_{j_2}\mathbb{1}_{\sigma_2})] = \\
&[\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}),\mathbb{1}_{\sigma_2}] - [\partial_{n_1+1}\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}),\mathrm{Cone}_{Y_2}(\partial_{j_2}\mathbb{1}_{\sigma_2})] = \\
&[\mathbb{1}_{\sigma_1},\mathbb{1}_{\sigma_2}] - ([\mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}),\mathbb{1}_{\sigma_2}] + [\partial_{n_1+1}\mathrm{Cone}_{Y_1}(\mathbb{1}_{\sigma_1}),\mathrm{Cone}_{Y_2}(\partial_{j_2}\mathbb{1}_{\sigma_2})]) = \\
&\mathbb{1}_{[\sigma_1,\sigma_2]} - \mathrm{Cone}_{Y_1*Y_2}(\partial_{j_1+j_2+1}\mathbb{1}_{[\sigma_1,\sigma_2]}).
\end{aligned}$$

Thus, the cone equation holds.

Let $0 \leq j \leq n_1 + n_2$. Then every $\sigma \in \overrightarrow{X}(j)$ is of the form $\sigma = [\sigma_1, \sigma_2]$ where $\sigma_i \in \overrightarrow{X}(j_i), i = 1, 2$ and $j_1 + j_2 + 1 = j$. Note that $j_2 \geq -1$ and thus $j_1 \leq j$. In particular, if $j < n_1$, then $j_1 < n_1$.

For such j_1, j_2, it holds that if $j_1 < n_1$, then

$$|\mathrm{supp}(\mathrm{Cone}_{Y_1*Y_2}(\mathbb{1}_{[\sigma_1,\sigma_2]}))| \leq R^{(1)}_{j_1}.$$

If $j_1 = n_1$, then $j_1 = j - n_1 - 1$ and

$$|\mathrm{supp}(\mathrm{Cone}_{Y_1*Y_2}(\mathbb{1}_{[\sigma_1,\sigma_2]}))| \leq |\mathrm{supp}(\mathbb{1}_{\sigma_1} - \mathrm{Cone}_{Y_1}(\partial_{n_1}\mathbb{1}_{\sigma_1}))| R^{(2)}_{j_2} \leq ((n_1+1)R^{(1)}_{n_1-1} + 1)R^{(2)}_{j-n_1-1}.$$

Combining these bounds yields the bound stated above for $\mathrm{Rad}_j(\mathrm{Cone}_{Y_1*Y_2})$. $\square$

3 Constructing a cone function be adding vertices

The idea is to construct a cone function for a given complex X, by starting with a given subcomplex for which we have a cone function and adding a set of vertices which is "well-behaved". The exact formalism of this idea is given in Theorem 3.2 that can be thought of as a special version of the Mayer-Vietoris theorem for cone functions.

Definition 3.1. *For a complex X, a subcomplex $X' \subseteq X$ is called full, if for every $v_0, \ldots, v_k \in X'$, if $\{v_0, \ldots, v_k\} \in X$, then $\{v_0, \ldots, v_k\} \in X'$.*

Theorem 3.2. *Let X be an n-dimensional simplicial complex and X' be a full subcomplex. Assume there is an $(n-1)$-cone function $\text{Cone}_{X'}$ and constants $R'_k \in \mathbb{N}, -1 \leq j \leq n-1$ such that for every $-1 \leq j \leq n-1$, $\text{Rad}_j(\text{Cone}_{X'}) \leq R'_j$. Also let $W \subseteq X(0)$ be a set of vertices such that the following holds:*

1. *$W \cap X' = \emptyset$.*
2. *For every $w_1, w_2 \in W$, $\{w_1, w_2\} \notin X(1)$.*
3. *For every $w \in W$, $X_w \cap X'$ is a non-empty simplicial complex.*
4. *There are $(n-2)$-cone functions $\text{Cone}_{X_w \cap X'}, w \in W$ and constants $R''_j \in \mathbb{N}, -1 \leq k \leq n-2$ such that for every $w \in W$ and every $-1 \leq j \leq n-2$, $\text{Rad}_j(\text{Cone}_{X_w \cap X'}) \leq R''_j$*

Let $X' \cup W$ be the full subcomplex of X spanned by $X'(0) \cup W$. Then there is an $(n-1)$-cone function $\text{Cone}_{X' \cup W}$ such that for every $-1 \leq j \leq n-1$, $\text{Rad}_j(\text{Cone}_{X' \cup W}) \leq R''_{j-1}(R'_j + 1)$.

Proof. We will define $\text{Cone}_{X' \cap W}$ by "adding" the cone functions of $X_w \cap X', w \in W$ to $\text{Cone}_{X'}$.

We note that by linearity and Observation 2.7, it is enough to define the cone function on every $\mathbb{1}_{[v_0,\ldots,v_j]}$ for every $0 \leq j \leq n-1$ and every $[v_0, \ldots, v_j] \in \overrightarrow{X' \cup W}(j)$. We also note that by our assumptions, every $[v_0, \ldots, v_j] \in \overrightarrow{X' \cup W}(j)$ is either in $\overrightarrow{X'}(j)$ or that it has a single vertex in W. Thus, it is enough to define the cone function on $\mathbb{1}_{[v_0,\ldots,v_j]}$ in the following two cases: either $[v_0, \ldots, v_j] \in \overrightarrow{X'}(j)$ or $v_0 = w \in W$ and $[v_1, \ldots, v_j] \in \overrightarrow{X_w \cap X'}(j-1)$.

After the preceding discussion, we define an $(n-1)$-cone function as a linear continuation of the following: First, we define $\text{Cone}_{X' \cup W}(\emptyset) = \mathbb{1}_{[v]}$ where v is the apex of the cone function $\text{Cone}_{X'}$. Second, for every $0 \leq j \leq n-1$ and every $[v_0, \ldots, v_j] \in \overrightarrow{X'}(j)$, we define

$$\text{Cone}_{X' \cup W}(\mathbb{1}_{[v_0,\ldots,v_j]}) = \text{Cone}_{X'}(\mathbb{1}_{[v_0,\ldots,v_j]}).$$

Last, for every $0 \leq j \leq n-1$, every $w \in W$ and every $[v_1, \ldots, v_j] \in \overrightarrow{X_w \cap X'}(j-1)$, we define

$$\text{Cone}_{X' \cup W}(\mathbb{1}_{[w,v_1,\ldots,v_j]}) = \text{Cone}_{X'}(\text{Cone}_{X_w \cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]})) - [w, \text{Cone}_{X_w \cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]})].$$

We need to verify that the cone equation holds for every $\mathbb{1}_{[w,v_1,\ldots,v_j]}$.

We will start by computing $\mathrm{Cone}_{X'\cup W}(\partial_j \mathbb{1}_{[w,v_1,\ldots,v_j]})$:

$$
\begin{aligned}
&\mathrm{Cone}_{X'\cup W}(\partial_j \mathbb{1}_{[w,v_1,\ldots,v_j]}) =^{(1)} \\
&\mathrm{Cone}_{X'\cup W}(\mathbb{1}_{[v_1,\ldots,v_j]}) - \mathrm{Cone}_{X'\cup W}([w, \partial_{j-1} \mathbb{1}_{[v_1,\ldots,.v_j]}]) = \\
&\mathrm{Cone}_{X'}(\mathbb{1}_{[v_1,\ldots,v_j]}) - \mathrm{Cone}_{X'}(\mathrm{Cone}_{X_w\cap X'}(\partial_{j-1} \mathbb{1}_{[v_1,\ldots,v_j]})) + \\
&[w, \mathrm{Cone}_{X_w\cap X'}(\partial_{j-1} \mathbb{1}_{[v_1,\ldots,v_j]})] =^{\text{the cone equation for } \mathrm{Cone}_{X_w\cap X'}} \\
&\mathrm{Cone}_{X'}(\mathbb{1}_{[v_1,\ldots,v_j]}) - \mathrm{Cone}_{X'}(\mathbb{1}_{[v_1,\ldots,v_j]}) + \mathrm{Cone}_{X'}(\partial_j \mathrm{Cone}_{X_w\cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]})) + \\
&[w, \mathrm{Cone}_{X_w\cap X'}(\partial_{j-1} \mathbb{1}_{[v_1,\ldots,v_j]})] = \\
&\mathrm{Cone}_{X'}(\partial_j \mathrm{Cone}_{X_w\cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]})) + [w, \mathrm{Cone}_{X_w\cap X'}(\partial_{j-1} \mathbb{1}_{[v_1,\ldots,v_j]})].
\end{aligned}
$$

Using this computation, we will verify the cone equation for $\mathrm{Cone}_{X'\cup W}$ (below we use the cone equation for $\mathrm{Cone}_{X'}$ and $\mathrm{Cone}_{X_w\cap X'}$ without noting it):

$$
\begin{aligned}
&\partial_{j+1} \mathrm{Cone}_{X'\cup W}(\mathbb{1}_{[w,v_1,\ldots,v_j]}) = \\
&\partial_{j+1} \mathrm{Cone}_{X'}(\mathrm{Cone}_{X_w\cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]})) - \partial_{j+1}[w, \mathrm{Cone}_{X_w\cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]})] = \\
&- \mathrm{Cone}_{X'}(\partial_j \mathrm{Cone}_{X_w\cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]})) + \mathrm{Cone}_{X_w\cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]}) - \\
&\Big(\mathrm{Cone}_{X_w\cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]}) - [w, \partial_j \mathrm{Cone}_{X_w\cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]})]\Big) = \\
&- \mathrm{Cone}_{X'}(\partial_j \mathrm{Cone}_{X_w\cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]})) - [w, \mathrm{Cone}_{X_w\cap X'}(\partial_{j-1} \mathbb{1}_{[v_1,\ldots,v_j]})] + [w, \mathbb{1}_{[v_1,\ldots,v_j]}] = \\
&- \mathrm{Cone}_{X'\cup W}(\partial_j \mathbb{1}_{[w,v_1,\ldots,v_j]}) + \mathbb{1}_{[w,v_1,\ldots,v_j]}
\end{aligned}
$$

as needed.

In order to conclude the proof, we will show that $\mathrm{Rad}_j^{n-1}(\mathrm{Cone}_{X'\cup W}) \leq R''_{j-1}(R'_j + 1)$. For $[v_0, \ldots, v_j] \in \overrightarrow{X'}(j)$,

$$
|\mathrm{supp}(\mathrm{Cone}_{X'\cup W}(\mathbb{1}_{[v_0,\ldots,v_j]}))| = |\mathrm{supp}(\mathrm{Cone}_{X'}(\mathbb{1}_{[v_0,\ldots,v_j]}))| \leq R'_k \leq R''_{j-1}(R'_j + 1).
$$

For $[w, v_1, \ldots, v_j] \in \overrightarrow{X' \cup W}(k)$ where $w \in W$,

$$
\begin{aligned}
&|\mathrm{supp}(\mathrm{Cone}_{X'\cup W}(\mathbb{1}_{[v_0,\ldots,v_j]}))| \leq \\
&|\mathrm{supp}(\mathrm{Cone}_{X'}(\mathrm{Cone}_{X_w\cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]})))| + |\mathrm{supp}([w, \mathrm{Cone}_{X_w\cap X'}(\mathbb{1}_{[v_1,\ldots,v_j]})])| \leq \\
&R'_j R''_{j-1} + R''_{j-1} = R''_{j-1}(R'_j + 1)
\end{aligned}
$$

as needed. □

4 The Kac–Moody–Steinberg complexes

In this section, we describe the families of simplicial complexes, for which we will prove that they give rise to cosystolic expanders. The main source, what we call KMS complexes, are infinite families of coset complexes over finite quotients of so called Kac–Moody–Steinberg groups. They where introduced in [GdPVB24], where it was shown that they are spectral high-dimensional expanders. Our methods also apply to the complexes introduced in [DLYZ23], since the links of these complexes are isomorphic to links of certain KMS complexes.

4.1 Root systems and Chevalley groups

We start by recalling some facts about generalized Cartan matrices, the associated root systems and Chevalley groups. These are the building blocks of KMS groups.

Most of the structure of the KMS groups is encoded in its generalized Cartan matrix, which also gives rise to a Dynkin diagram and a root system.

Definition 4.1. *A generalized Cartan matrix (GCM) is a matrix $A = (A_{ij})_{i,j \in I} \in \mathrm{Mat}_n(\mathbb{Z})$ such that $A_{ii} = 2$ for all $i \in I$, $A_{ij} \leq 0$ for all $i \neq j \in I$ and $A_{ij} = 0 \iff A_{ji} = 0$.*

Every GCM $A = (A_{i,j})_{i,j \in I}$ gives rise to a Dynkin diagram in the following way. As vertex set, we take the index set I and two vertices i, j are connected by $|A_{i,j}|$ edges if $A_{i,j}A_{j,i} \leq 4$ and $|A_{i,j}| \geq |A_{j,i}|$, and these edges are equipped with an arrow pointing towards i if $|A_{i,j}| > 1$. If $A_{i,j}A_{j,i} > 4$, the vertices i and j are connected by a bold-faced labelled edge with the ordered pair of integers $|A_{i,j}|, |A_{i,j}|$.

A GCM is irreducible *if there exists no non-trivial partition $I = I_1 \cup I_2$ such that $A_{i_1,i_2} = 0$ for all $i_1 \in I_1, i_2 \in I_2$.*

Definition 4.2. *Let A be a $(d+1)\times(d+1)$ generalized Cartan matrix with index set $I = \{0, \dots, d\}$, and let $J \subseteq I$.*

(i) The subset J is called spherical *if $A_J = (A_{i,j})_{i,j \in J}$ is of spherical type, meaning that the associated Coxeter group (see e.g. [Mar18, Proposition 4.22]) is finite (see e.g. [Bou08, Chapter 6.4.1]). Given $n \geq 2$, A is n-*spherical *if every subset $J \subseteq I$ of size n is spherical.*

(ii) We denote by Q_A the set of spherical subsets of I associated to the generalized Cartan matrix A.

(iii) A generalized Cartan matrix A is purely n-spherical *if every spherical subset $J \subseteq I$ is contained in a spherical subset of size n. (In particular, no set of size $n+1$ is spherical for a purely n-spherical generalized Cartan matrix.)*

(iv) Analogously, we call a GCM n-classical *if each irreducible factor of $A_J = (A_{i,j})_{i,j \in J}$ is of classical type A_k, B_k, C_k, D_k (see e.g. [Bou08, Chapter 6.4.1]) for each $J \subseteq I$ with $|J| \leq n$.*

(v) To a generalized Cartan matrix A we can associate the following sets:

- *a set of simple roots $\Pi = \{\alpha_i \mid i \in I\}$,*
- *a set of real roots $\Phi \subseteq \bigoplus_{i \in I} \mathbb{Z}\alpha_i$,*
- *two sets, one of positive and one of negative real roots $\Phi^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i \cap \Phi, \Phi^- = -\Phi^+$.*

Note that $\Phi = \Phi^+ \sqcup \Phi^-$. More details can be found e.g. in [Mar18, Chapter 3.5].

Remark 4.3. *(i) Irreducible finite root systems have been classified. All possible diagrams are given in Figure 1.*

(ii) The n-classical diagrams, with $n \geq 3$ are precisely the non-spherical rank $n+1$ diagrams which are n-spherical but not one of the following

- *affine diagrams $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8, \tilde{F}_4, \tilde{G}_2$,*

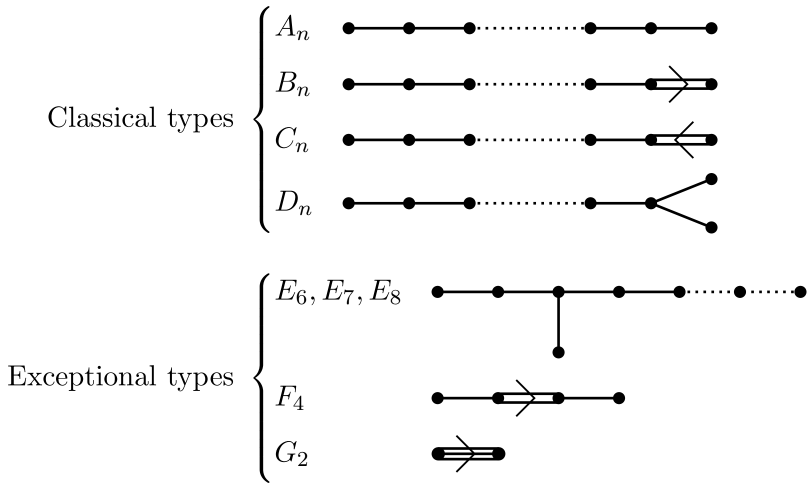
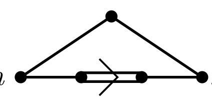


Figure 1: The Dynkin diagrams of irreducible spherical root systems

- *compact hyperbolic diagram*  .

All rank $n+1$, n-spherical diagrams have been classified and are precisely the affine diagrams and the compact hyperbolic diagrams, the latter have rank at most 5, where the diagram of rank 5 is exactly the one that we have to exclude to get n-classicality. See e.g. [Hum90, Chapter 6.9] for the classification result.

Definition 4.4. *Corresponding to any irreducible, spherical GCM $\mathring{A}$ with root system $\mathring{\Phi}$ of rank at least 2, and any finite field K, there is an associated universal (or simply connected) Chevalley group, denoted* $\mathrm{Chev}_{\mathring{A}}(K)$. *Abstractly, it is generated by symbols $x_\alpha(s)$ for $\alpha \in \mathring{\Phi}$ and $s \in K$, subject to the relations*

$$\begin{aligned}
x_\alpha(s)x_\alpha(u) &= x_\alpha(s+u) \\
[x_\alpha(s), x_\beta(u)] &= \prod_{i,j>0} x_{i\alpha+j\beta}\left(C_{ij}^{\alpha,\beta} s^i u^j\right) \quad (\textit{for } \alpha+\beta \neq 0) \\
h_\alpha(s)h_\alpha(u) &= h_\alpha(su) \quad (\textit{ for } s,u \neq 0), \\
\textit{where} \quad h_\alpha(s) &= n_\alpha(s)n_\alpha(-1) \\
\textit{and} \quad n_\alpha(s) &= x_\alpha(s)x_{-\alpha}\left(-s^{-1}\right)x_\alpha(s).
\end{aligned}$$

Note that the $C_{ij}^{\alpha,\beta}$ are integers called structure constants that can be found in [Car89].

Remark 4.5. *Let K be a field and let $K[t]$ denote the polynomial ring in one variable over K. The simply-connected Chevalley group* $\mathrm{Chev}_{\mathring{A}}(K[t])$ *is, similar to the case of a Chevalley group over a field, generated by elements $x_\alpha(s)$ for $\alpha \in \mathring{\Phi}, s \in K[t]$ that satisfy the relations above, where for the third relation we have to add the extra assumption that u, s are invertible in $K[t]$. This can be found e.g. in [Reh75].*

4.2 Kac–Moody–Steinberg groups

We can now put the ingredients together to define what a Kac–Moody–Steinberg group is and present some of its properties. For more details, see [GdPVB24, Chapter 3] and references therein.

Definition 4.6. *Let K be a field. Let $A = (A_{i,j})_{i,j\in I}$ be a GCM over the index set I which is 2-spherical. Let $Q_A = \{J \subseteq I \mid A_J := (A_{ij})_{i,j\in J}$ is spherical$\}$. Let Φ be the root system associated to A with simple roots $\Pi = \{\alpha_i \mid i \in I\}$. For $J \in Q_A$ set*

$$U_J := \langle x_{\alpha_i}(s) \mid s \in K, i \in J\rangle \leq \mathrm{Chev}_{A_J}(K)$$

the group generated by all simple roots in the Chevalley group of type A_J over K. Note that if $L \subset J$ then we have a natural inclusion $U_L \hookrightarrow U_J$ by sending $x_{\alpha_i}(s) \in U_L$ to the same generator in U_J. The KMS group of type A over K is defined as the free product of the $U_J, J \in Q_A$ modulo the natural inclusions:

$$\mathcal{U}_A(K) = \mathop{\ast}_{J\in Q_A} U_J/(U_L \hookrightarrow U_J, L \subseteq J).$$

Note that $U_J \hookrightarrow \mathcal{U}_A(K)$ and we will denote the image of U_J in $\mathcal{U}_A(K)$ again by U_J. These subgroups are called the local groups of $\mathcal{U}_A(K)$. For $i, j \in I$ we will write $U_i := U_{\{i\}}$ and $U_{i,j} = U_{\{i,j\}}$.

Remark 4.7. *For two roots $\alpha, \beta \in \Phi$ we write $]\alpha, \beta[_{\mathbb{N}}= \{n_1\alpha + n_2\beta \in \Phi \mid n_1, n_2 \in \mathbb{N}^*\}$ and $[\alpha, \beta]_{\mathbb{N}} =]\alpha, \beta[_{\mathbb{N}} \cup \{\alpha, \beta\}$.*

We have the following abstract presentation for the KMS groups:

$$\mathcal{U}_A(K) = \left\langle u_\beta(t) \text{ for } t \in K, i, j \in I, \beta \in [\alpha_i, \alpha_j]_{\mathbb{N}} \mid \mathcal{R}\right\rangle$$

where the set of relations $\mathcal{R}$ is defined as:

for all $i, j \in I$, $\{\alpha, \beta\} \subseteq [\alpha_i, \alpha_j]_{\mathbb{N}}$, $t, u \in K$:

$$u_\alpha(t)u_\alpha(s) = u_\alpha(t + s)$$
$$[u_\alpha(t), u_\beta(u)] = \prod_{\gamma=k\alpha+l\beta\in]\alpha,\beta[_{\mathbb{N}}} u_\gamma\left(C^{\alpha\beta}_{k,l} t^k u^l\right).$$

The constants $C^{\alpha,\beta}_{k,l}$ are the same structure constants as in the presentation of the Chevalley group.

Since A is 2-spherical, the subgroups U_β, for $\beta \in [\alpha_i, \alpha_j]$, are contained in the group generated by the root groups $U_{\alpha_i}, U_{\alpha_j}$ (see [Abr96, Proposition 7]).

4.3 KMS complexes

In this section, we recall some facts about coset complexes and put everything together to define KMS complexes.

KMS complexes are coset complexes. More details on coset complexes can be found for example in [KO23], and we will only give a brief overview.

Definition 4.8. *Let G be a group and $\mathcal{H} = (H_0, \ldots, H_n)$ be a family of subgroups of G. Then the coset complex $\mathcal{CC}(G; (H_0, \ldots, H_n))$ is defined to be the simplicial complex with*

- *vertex set* $\bigsqcup_{i=0}^{n} G/H_i$
- *a set of vertices* $\{g_1 H_{i_1}, \ldots, g_k H_{i_k}\}$ *forms a* $(k-1)$*-simplex in* $\mathcal{CC}(G;\mathcal{H})$ *if and only if* $\bigcap_{j=1}^{k} g_j H_{i_j} \neq \emptyset$.

Note that the set of vertices is partitioned into $n+1$ subsets G/H_i for $i = 0, \ldots, n$ such that for any simplex $\sigma \in \mathcal{CC}(G;\mathcal{H})$ we have $|\sigma \cap G/H_i| \leq 1$ for all i. In that case, we say that $\mathcal{CC}(G;\mathcal{H})$ is a $(n+1)$-partite simplicial complex.

Definition 4.9. *The* type *of a simplex* $\{g_1 H_{i_1}, \ldots, g_k H_{i_k}\}$ *in* $\mathcal{CC}(G;\mathcal{H})$ *is the set of indices* $\{i_1, \ldots, i_k\} \subseteq \{0, \ldots, n\}$. *Given a type* $\emptyset \neq T \subseteq \{0, \ldots, n\}$ *we write*

$$H_T := \bigcap_{i \in T} H_i \qquad \text{and set} \qquad H_\emptyset := \langle H_0, \ldots, H_n \rangle \leq G.$$

The following well-known fact will be very useful.

Proposition 4.10. *Let* σ *be a face in* $\mathcal{CC}(G,\mathcal{H})$ *of type* $T \neq \emptyset$. *Then the link of* σ *is isomorphic to the coset complex* $\mathcal{CC}\left(H_T, \left(H_{T\cup\{i\}} : i \notin T\right)\right)$.

KMS complexes are coset complexes over certain finite quotients of KMS groups. The precise definition is as follows.

Definition 4.11. *Let* A *be a GCM over the index set* I *which is* $(|I|-1)$*-spherical but non-spherical. Let* K *be a finite field of size* $q \geq 4$. *Let* $\phi : \mathcal{U}_A(K) \to G$ *be a finite quotient of* $\mathcal{U}_A(K)$ *such that*

1. $\phi|_{U_J}$ *is injective for all* $J \in Q_A$ *(i.e.* ϕ *is injective on the local groups),*
2. $\phi(U_J \cap U_L) = \phi(U_J) \cap \phi(U_L)$ *for all* $J, L \in Q_A$.

Then we set

$$X = \mathcal{CC}(G; (\phi(U_{I\setminus\{j\}}))_{j\in I})$$

and call X *a KMS complex of type* A *over* K.

Theorem 4.12. *[GdPVB24, Theorem 4.3] Let* A *be a GCM over the index set* I *which is non-spherical but* $|I|-1$*-spherical. Let* K *be a finite field of size* $q \geq 4$. *Let* $\phi_i : \mathcal{U}_A(K) \to G_i, i \in I$ *be family of finite quotients of* $\mathcal{U}_A(K)$ *such that* $X_i = \mathcal{CC}(G_i, (\phi_i(U_{I\setminus\{j\}}))_{j\in I}), i \in \mathbb{N}$ *is a family of KMS complexes and* $|G_i| \xrightarrow{i\to\infty} \infty$. *If* q *is such that* $\sqrt{\frac{3}{q}} \leq \frac{1}{|I|-1}$ *then the family* $(X_i)_{i\in\mathbb{N}}$ *is a family of bounded degree local spectral expanders.*

Moreover, if $q \gg (|I|-1)^2$, *then the family* $(X_i)_{i\in\mathbb{N}}$ *is a family of bounded degree* $\frac{2}{\sqrt{q}}$*-local spectral expanders.*

Example 4.13. The following is an example of an n-dimensional, n-classical KMS complex. Consider $\mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ where $n \geq 2$ and q is an odd prime power such that $q \geq 5$. For $1 \leq i, j \leq n+1, i \neq j$ and $f \in \mathbb{F}_q[t]$, we denote $e_{i,j}(f) \in \mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ to be the matrix with 1's along the main diagonal, f in the (i,j)-th entry and 0's in all other entries. Let

$$B = \{e_{i,i+1}(1); i = 1, \ldots n\} \cup \{e_{n+1,1}(t)\}$$

and let $\mathcal{U}(\mathbb{F}_q) < \mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ be the subgroup generated by B, i.e.

$$\mathcal{U}(\mathbb{F}_q) = \langle e_{1,2}(1), e_{2,3}(1), \dots, e_{n,n+1}(1), e_{n+1,1}(t)\rangle.$$

We define subgroups $H_0, \dots, H_n < \mathcal{U}(\mathbb{F}_q)$ as

$$\begin{aligned}
H_0 &= \langle e_{1,2}(1), e_{2,3}(1), \dots, e_{n,n+1}(1)\rangle = \langle B \setminus \{e_{n+1,1}(t)\}\rangle,\\
H_1 &= \langle e_{2,3}(1), \dots, e_{n,n+1}(1), e_{n+1,1}(t)\rangle = \langle B \setminus \{e_{1,2}(1)\}\rangle,\\
&\ \vdots\\
H_i &= \langle B \setminus \{e_{i,i+1}\}\rangle,\\
&\ \vdots\\
H_n &= \langle e_{1,2}(1), e_{2,3}(1), \dots, e_{n-1,n}(1), e_{n+1,1}(t)\rangle,
\end{aligned}$$

and denote $\mathcal{H} = (H_0, \dots, H_n)$.

Let $f \in \mathbb{F}_q[t]$ be an irreducible polynomial of degree $s > 1$. Denote $\phi_f : \mathrm{SL}_{n+1}(\mathbb{F}_q[t]) \to \mathrm{SL}_{n+1}(\mathbb{F}_q[t]/(f)) \cong \mathrm{SL}_{n+1}(\mathbb{F}_{q^s})$ defined as follows: For every matrix $A \in \mathrm{SL}_{n+1}(\mathbb{F}_q[t]), A = (A_{i,j})$ where $A_{i,j} \in \mathbb{F}_q[t]$ define

$$\phi_f(A) = (A_{i,j} + (f)).$$

Restricting ϕ_f to $\mathcal{U}(\mathbb{F}_q)$, we get $G_f = \phi_f(\mathcal{U}(\mathbb{F}_q))$ which is a finite quotient of $\mathcal{U}(\mathbb{F}_q)$. It is proven in [GdPVB24], that ϕ_f is injective on $H_0, \dots, H_n$. The KMS complex of $G_f = \phi_f(\mathcal{U}(\mathbb{F}_q))$ is defined to be the coset complex $X_f = \mathcal{CC}(G_f, \mathcal{H}_f)$, where $\mathcal{H}_f = (\phi_f(H_0), \dots, \phi_f(H_n))$.

Remark 4.14. *Similar examples of n-classical KMS complexes can be constructed for all Chevalley groups of classical type, see [GdPVB24, Chapter 5] for more details.*

5 Describing classical buildings and their opposition complexes

In the first part of this section, we describe buildings and their opposition complexes from a group theoretic view point. In this setting, we show that the links of KMS complexes are opposition complexes of spherical buildings. In the second part, we describe a more geometric view point using flag complexes. This context will be used to show that opposition complexes are coboundary expanders.

5.1 Buildings and BN-pairs

One way to define a building is as a simplicial complex with a family of subcomplexes called apartments. Each apartment isomorphic to the same Coxeter complex. The type of the Coxeter complex, which can be described by a root system/ a GCM, is also what we call the type of the building. Each two simplices of the building have to be contained in a common apartment and each two apartments are isomorphic with an isomorphism that acts as the identity on the intersection.

One way to construct buildings is via groups with BN-pair (and for thick, irreducible buildings of dimension 2 and larger, all buildings can be constructed that way, as was proven by Jacques Tits in [Tit74]).

Given a group G, a BN-pair of G is a pair of subgroups $B, N \leq G$ satisfying certain properties, see [AB08, Definition 6.55]. In particular, $G = BN$ and setting $T = B \cap N$ we have that $W = N/T$

admits a set of generators S, such that (W, S) is a Coxeter system. W is also called the Weyl group of the BN-pair. To describe the building, we need to define the parabolic subgroups $P_J := B\langle J\rangle B$ for $J \subset S$ (here $\langle J\rangle$ is the subgroup generated by the elements of J inside W). The building can then we described as the following coset complex:

$$\Delta(G, B) := \mathcal{CC}(G; (P_{S\setminus\{j\}})_{j\in S}).$$

Since $P_J \cap P_L = P_{J\cap L}$ for any $J, L \subseteq S$ we can see $\Delta(G, B)$ also as the poset $\{gP_J \mid g \in G, J \subseteq S\}$ ordered by reversed inclusion. Then the simplex $\{gP_{S\setminus\{j\}} \mid j \in J\} \in \mathcal{CC}(G; (P_{S\setminus\{j\}})_{j\in S})$ corresponds to $gP_{S\setminus J}$ for $J \subseteq S, g \in G$. In particular, the set of maximal simplices, called chambers, corresponds to G/B.

If the Weyl group is finite, we call the building spherical. This coincides with our definition of calling a GCM/root system spherical. A finite Weyl group has a unique longest element denoted by w_0, where longest means that is generated by the largest number of generators, counted with multiplicities.

Two chambers $gB, hB, g, h \in G$ are called opposite if and only if $g^{-1}h \in Bw_0B$, denoted by gB op hB.

Given a simplex $\sigma \in \Delta(G, B)$, the complex opposite σ is defined as follows:

$$\Delta^0(\sigma) := \{\tau \in \Delta(G, B) \mid \text{ there exist chambers } c, d \in \Delta(G, B) \text{ such that } \tau \subseteq c, \sigma \subseteq d, c \text{ op } d\}.$$

To see the connection between opposition complexes and the links of KMS complexes, note that every Chevalley group has a BN-pair in the following way. Let $A = (A_{ij})_{i,j\in I}$ be a spherical GCM of rank at least 2, K a field and $G = \mathrm{Chev}_A(K) = \langle x_\alpha(\lambda) \mid \alpha \in \Phi, \lambda \in K\rangle$, using the notation of Definition 4.4. We set

$$\begin{aligned} &U_\alpha := \langle x_\alpha(\lambda) \mid \lambda \in K\rangle, \alpha \in \Phi, \qquad U^+ := \langle U_\alpha \mid \alpha \in \Phi^+\rangle, \qquad T := \langle h_\alpha(\lambda) \mid \alpha \in \Phi, \lambda \in K\rangle, \\ &B := U^+T, \qquad\qquad\qquad\qquad\qquad N := \langle T, n_{-\alpha_i}(-\lambda^{-1}); i \in I, \lambda \in K^*\rangle. \end{aligned}$$

For example [AB08, Theorem 7.115] shows that this is indeed a BN-pair.

Let $C_0 = 1B$ denote the fundamental chamber of $\Delta(G, B)$ where G is a Chevalley group as described above. Recall that w_0 denotes the longest element on the Weyl group W. It satisfies $Bw_0B = U^+w_0B$. We can describe the opposition complex in this set-up as

$$\begin{aligned} \Delta^0(C_0) &= \{gP_J \mid g \in G, J \subseteq S : \exists h \in U^+w_0B : gP_J \supseteq hB\} \\ &= \{gP_J \mid \exists h \in U^+w_0B : h \in gP_J\} \\ &= \{hP_J \mid h \in U^+w_0B\} \\ &= \{uw_0P_J \mid u \in U^+\}. \end{aligned}$$

This leads to the following proposition.

Proposition 5.1. *[GdPVB24, Chapter 3.2] Let G be a Chevalley group with BN-pair as described above. Then*

$$\mathcal{CC}(U^+, (U_{I\setminus\{i\}})_{i\in I}) \cong \Delta^0(C_0).$$

Since we assume that A is 2-spherical and $|K| \geq 4$ we have $U^+ = \langle U_\alpha \mid \alpha \in \Phi^+\rangle = \langle U_{\alpha_i} \mid i \in I\rangle$ (see e.g. the comment before [GdPVB24, Lemma 3.7]). Combining this with the above proposition and Proposition 4.10 gives the following corollary.

Corollary 5.2. *Let X be a KMS complex, and $\sigma \in X$ a face of dimension less than or equal to $\dim X - 2$ of type $\emptyset \neq T \subset I$. Then*

$$\mathrm{lk}_X(\sigma) \cong \Delta^0(C_0)$$

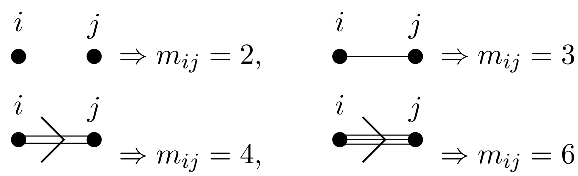
where Δ is the building obtained from the BN-pair structure of the Chevalley group of type $A_{I \setminus T}$.

5.2 Types of buildings

To each Dynkin diagram with set of vertices I as in Definition 4.1 we can associate a group, called Coxeter groups, in the following way:

$$\langle s_i; i \in I \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1; i, j \in I, i \neq j \rangle$$

where



Note that m_{ij} is defined to be symmetric in i, j, this implies that the root systems of type B_n and C_n give rise to the same Coxeter group. The Coxeter complex associated to the Coxeter group W with set of generators S can be viewed as the coset complex

$$\mathcal{CC}(W, (\langle S \setminus \{t\} \rangle)_{t \in S}).$$

Thus, a building of type X means that we consider a building where the underlying Coxeter complex is of type X. In particular, since type B_n and C_n give rise to the same Coxeter complex, they also determine the same type of building.

Note that if we construct a building from a BN-pair of a Chevalley group with underlying root system of some type X, then the building will have the same type.

More details can be found in Tits original work on the classification of spherical buildings [Tit74] or for example in [AB08].

5.3 Geometric constructions of buildings

In the following section, we describe how buildings of type A_n, C_n and D_n can be constructed as flag complexes of certain sets equipped with an incidence relation. We furthermore describe the opposition complexes in this set-up. The general framework is as follows.

Definition 5.3. *Let X be a set and let $I \subseteq X \times X$ be a reflexive and symmetric relation. For $a, b \in X$ we write aIb if and only if $(a, b) \in I$ and call a and b incident. The structure gives rise to the following simplicial complex, called the flag complex of X:*

$$\mathrm{Flag}(X) = \{\sigma \subseteq X \mid \forall a, b \in \sigma : aIb\}.$$

5.3.1 Buildings of type A_n

Let $n \in \mathbb{N}, n \geq 1$. Any building of type A_n can be described in the following way, see e.g. [Tit74]. Let K be a field or skew field, V an $(n+2)$-dimensional vector space over K. Set $X := X(V) := \{0 < U < V \mid U \text{ is a } K\text{-subspace of } V\}$. Two subspaces $U, W \in X$ are called incident if $U \subseteq W$ or $W \subseteq U$. Then $\Delta = \operatorname{Flag} X$ is a building of type A_n.

To describe the opposition complex in this context, we need the following notation, see [Abr96, Definition 9].

Definition 5.4. *1. Two subspaces U, W of V are called transversal (in V) if $U \cap W = 0$ or $U + W = V$. If this is the case, we write $U \pitchfork_V W$ or simply $U \pitchfork W$.*

2. Let $\mathcal{E}$ be a set of subspaces of V. Then we write $U \pitchfork \mathcal{E}$ if $U \pitchfork E$ for all $E \in \mathcal{E}$. We set

$$X_{\mathcal{E}}(V) := \{U \in X \mid U \pitchfork \mathcal{E}\} \qquad T_{\mathcal{E}}(V) = \operatorname{Flag} X_{\mathcal{E}}(V).$$

We get the following result.

Proposition 5.5. *[Abr96, Corollary 12] For any simplex $\sigma = \{E_1 < \cdots < E_r\} \in \Delta$ set $\mathcal{E}(\sigma) = \{E_i \mid 1 \leq i \leq r\}$. Then*

$$\Delta^0(\sigma) = T_{\mathcal{E}(\sigma)}(V).$$

Definition 5.6. *We denote by $\mathbf{C}_A$ the class of all simplicial complexes $T_{\mathcal{E}}(V)$ where $\dim(T_{\mathcal{E}}(V)) = n \geq 0$, K is a (skew) field, V an $(n+2)$-dimensional vector space over K, $\mathcal{E}$ is a finite set of subspaces of V such that if we set $e_j = |\{E \in \mathcal{E} \mid \dim E = j\}|$ we have $\sum_{j=1}^{n+1} \binom{n}{j-1} e_j \leq |K|$. Furthermore, $T_{\mathcal{E}}(V)$ is defined as described in Definition 5.4.*

5.3.2 Hermitian and pseudo-quadratic forms

In this part, we recall facts about hermitian and pseudo-quadratic forms that are needed to describe the buildings of type C_n and D_n. We follow [Abr96, Chapter 5], who in turn mostly follows [Tit74].

Let K be a skew field, $\sigma : K \to K, a \mapsto a^\sigma$ be an involution, i.e. an anti-automorphism of K such that $\sigma^2 = \mathrm{id}_K$. Let $\varepsilon \in \{1, -1\} \subset K$. If $\sigma \neq \mathrm{id}_K$ then we require $\varepsilon = -1$. Let V be a right K-vector space of dimension $m \in \mathbb{N} \cup \{\infty\}$. Let $K_{\sigma,\varepsilon} := \{\alpha - \alpha^\sigma \varepsilon \mid \alpha \in K\}$ and $K^{\sigma,\varepsilon} := \{\alpha \in K \mid \alpha + \alpha^\sigma \varepsilon = 0\}$. Let Λ be a form parameter relative to (σ, ε), i.e. Λ is a subgroup of $(K, +)$ satisfying $K_{\sigma,\varepsilon} \subseteq \Lambda \subseteq K^{\sigma,\varepsilon}$ and $\alpha^\sigma \Lambda \alpha \subseteq \Lambda$ for all $\alpha \in K$.

Let $f : V \times V \to K$ be a (σ, ε)-hermitian form, i.e. f is biadditive and for all $x, y \in V, a, b \in K$ we have $f(xa, yb) = a^\sigma f(x, y) b$ and $f(y, x) = f(x, y)^\sigma \varepsilon$. Let $Q : V \to K/\Lambda$ be a (σ, ε)-quadratic form with associated (σ, ε)-hermitian form f, i.e. $Q(xa) = a^\sigma Q(x) a + \Lambda$ and $Q(x + y) - Q(x) - Q(y) = f(x, y) + \Lambda$ for all $x, y \in V, a \in K$. If $\Lambda = K$ we require that f is alternating. If $\Lambda \neq K$ then f is uniquely defined by Q.

For $M \subseteq V$ we write $M^\perp := \{x \in V \mid f(x, M) = 0\}$. A subspace $U < V$ is called

- non-degenerate if $U \cap U^\perp = 0$,
- totally degenerate if $U \subseteq U^\perp$,
- anisotropic if $0 \notin Q(U \setminus \{0\})$,
- isotropic if $0 \in Q(U \setminus \{0\})$,

- totally isotropic if $U \subseteq U^\perp$ and $Q(U) = 0$.

We require that there is at least one finite-dimensional maximal totally isotropic subspace. In that case, all maximal totally isotropic subspace have the same dimension. We denote this dimension by n and call it the Witt index of (V, Q, f). We require $0 < n < \infty$. Furthermore, we require that $V^\perp = 0$. We say that the triple (V, Q, f) is a pseudo-quadratic space if it is of the form described above.

If it additionally satisfies $(m, \Lambda) \neq (2n, 0)$ then we say (V, Q, f) is a thick pseudo-quadratic space.

Note that if (V, Q, f) is a thick pseudo-quadratic space and K is a finite field, then $\dim V \leq 2n + 1$. In general, if one only considers finite fields, especially of characteristic different from 2, the set-up can be substantially simplified, see [AB08, Remark 9.3,9.4].

5.3.3 Buildings of type C_n

Let $n \in \mathbb{N}, n \geq 1$ and (V, Q, f) be a thick pseudo-quadratic space with Witt index n. Set $X := X(V) = \{0 < U < V \mid U \text{ is totally isotropic}\}$ with incidence relation given by containment as in the A_n case and $\Delta = \operatorname{Flag} X$. Then Δ is a thick building of type C_n and every classical C_n building can be obtained in this way (here classical means that the links of type A_2 correspond to Desarguesian planes, which is always the case for $n \geq 4$), see [Tit74, Theorem 8.22].

Definition 5.7. *Set $\mathcal{U} := \{0 \leq U \leq V \mid \dim U < \infty\}, \mathcal{U}^\perp := \{U^\perp \mid u \in \mathcal{U}\}$ and $\mathcal{W} := \mathcal{U} \cup \mathcal{U}^\perp$.*

Let $\mathcal{E}$ be a finite subset of $\mathcal{W}$ such that $\mathcal{E}^\perp = \mathcal{E}$. Assume $K = \mathbb{F}_q$ is a finite field. Set $\mathcal{E}_j := \{E \in \mathcal{E} \mid \dim E = j\}, e_j := |\mathcal{E}_j|$ and $e_h^{(s)} := \sum_{j=0}^{2s} \binom{2s}{j} e_{h+j}$ for $h \in \mathbb{N}, s \in \mathbb{N}_0$ and $h + 2s < m$. We define

- $N(\mathcal{E}) := e_1^{(n-1)}$ *if f is alternating ($\Rightarrow m = 2n$) and $Q = 0$,*
- $N(\mathcal{E}) := \left(e_1^{(n-1)}\right)^2$ *if $m = 2n$ and $\sigma \neq \mathrm{id}$,*
- $N(\mathcal{E}) := 2e_2^{(n-1)}$ *if $m = 2n + 1$,*
- $N(\mathcal{E}) := \max\left\{e_2^{(n-1)} + e_3^{(n-1)} + 1, 2e_3^{(n-1)}\right\}$ *if $m = 2n + 2$.*

Definition 5.8. *For a set $\mathcal{E}$ of subspaces of V we define, similar to Definition 5.4,*

$$X_{\mathcal{E}}(V) := \{U \in X \mid U \pitchfork \mathcal{E}\}, \qquad T_{\mathcal{E}}(V) = \operatorname{Flag} X_{\mathcal{E}}(V).$$

Proposition 5.9. *[Abr96, Corollary 15] For any simplex $\tau = \{E_1 < \cdots < E_r\} \in \Delta$ set $\mathcal{E}(\tau) = \{E_i, E_i^\perp \mid 1 \leq i \leq r\}$. Then $\Delta^0(\tau) = T_{\mathcal{E}(\tau)}(V)$.*

Definition 5.10. *With $\mathbf{C}_C$ we denote the class of simplicial complexes $T_{\mathcal{E}}(V)$ where $n = \dim(T_{\mathcal{E}}(V) + 1 \geq 1$, K a (skew) field, (V, Q, f) is a thick pseudo-quadratic space with Witt index n, $\mathcal{E}$ a finite subset of $\mathcal{W}$ (see Definition 5.7) such that $\mathcal{E}^\perp = \mathcal{E}$. If K is a finite field, we further require that $|K| \geq N(\mathcal{E})$. We set $T_{\mathcal{E}}(V)$ as in Definition 5.8.*

5.3.4 Buildings of type D_n

For this section let (V, Q, f) be a pseudo-quadratic space but in the specific case where K is a field, $\sigma = \mathrm{id}_K, \varepsilon = 1, \Lambda = 0$, V a K-vector space of dimension $m = 2n \geq 4$, Q an ordinary quadratic form and f the corresponding symmetric bilinear form. We further assume that V is non-degenerate and of Witt index n.

We set as before $X = X(V) = \{0 < U < V \mid U \text{ is totally isotropic}\}$. Then $\Delta = \operatorname{Flag} X$ is a weak C_n building, in particular it is not thick, since each totally isotropic subspace of dimension $n-1$ is contained in exactly two totally isotropic subspaces of dimension n. Furthermore, the space $Y = \{U \in X \mid \dim U = n\}$ is partitioned into to sets $Y = Y_1 \sqcup Y_2$ such that for $U_1, U_2 \in Y_i$ we have $n - \dim(U_1 \cap U_2) \in 2\mathbb{Z}$ for $i = 1, 2$ and for $U_1 \in Y_1, U_2 \in Y_2$ we have $n - \dim(U_1 \cap U_2) \in 2\mathbb{Z} + 1$. In particular, if for $U_1, U_2 \in Y$ we have $\dim(U_1 \cap U_2) = n - 1$ then U_1 and U_2 are in different sets of the partition.

To obtain a thick building of type D_n we set $\tilde{X} = \tilde{X}(V) = \{U \in X \mid \dim U \neq n - 1\}$ and we say that $U, W \in \tilde{X}$ are incident (write UIW) if and only if

$$U \subseteq W \text{ or } W \subseteq U \text{ or } \dim(W \cap U) = n - 1.$$

Then $\tilde{\Delta} = \operatorname{Flag} \tilde{X}$ is desired building. We will write $\operatorname{Orifl}(\tilde{X})$ for $\operatorname{Flag} \tilde{X}$ to stress that we consider $\tilde{X}$ not with the usual incidence relation given by inclusion, but with this new one. Note that $\operatorname{Orifl} \tilde{X}$ is called the oriflamme complex of $\tilde{X}$.

For $n \geq 4$ every building of type D_n is of the form $\tilde{\Delta}$ for some field K [Tit74, Proposition 8.4.3]. We also consider the construction for $n = 2, 3$ in which case we get certain (but not all) buildings of type $D_2 = A_1 \times A_1$ and $D_3 = A_3$, which where already covered earlier.

To describe the opposition complexes, we need to introduce some further notation.

Definition 5.11. *For arbitrary subspaces U, W of V, we define*

$$U \tilde{\pitchfork}_V W \iff U \pitchfork_V W \text{ or } (U, W \in \tilde{X}, U = U^\perp, W = W^\perp \text{ and } \dim(U \cap W) = 1).$$

For a set $\mathcal{E}$ of subsets of V we define (different from Definition 5.4)

$$X_{\mathcal{E}}(V) = \{U \in X \mid U \tilde{\pitchfork} \mathcal{E}\}, \qquad T_{\mathcal{E}}(V) = \operatorname{Flag} X_{\mathcal{E}}(V)$$
$$\tilde{X}_{\mathcal{E}}(V) = \{U \in \tilde{X} \mid U \tilde{\pitchfork} \mathcal{E}\}, \qquad \tilde{T}_{\mathcal{E}}(V) = \operatorname{Orifl} \tilde{X}_{\mathcal{E}}(V)$$

Proposition 5.12. *[Abr96, Corollary 17] Let $\tau = \{E_1 < \cdots < E_r\} \in \tilde{\Delta}$ and set $\mathcal{E}(\tau) = \{E_i, E_i^\perp \mid 1 \leq i \leq r\}$ then $\tilde{\Delta}^0(\tau) = \tilde{T}_{\mathcal{E}(\tau)}(V)$.*

6 Coboundary expansion for sub-complexes of spherical buildings

6.1 General induction argument

The general procedure will be similar for all three cases of buildings. We use a quantitative version of [Abr96, Lemma 22]. We basically only change assumption 1. and the conclusion.

Theorem 6.1. *Let $\mathbf{C}$ be a class of non-empty simplicial complexes. Assume that for any $n \in \mathbb{N}, n \geq 1$ there exists $\ell_n \in \mathbb{N}$ such that for all $\kappa \in \mathbf{C}, \dim \kappa = n$, there exists a filtration $\kappa_0 \subset \kappa_1 \subset \ldots \subset \kappa_\ell = \kappa$ of κ of length $\ell \leq \ell_n$ such that the following is satisfied (set $V_i := \{\text{vertices of } \kappa_i\} \setminus \{\text{vertices of } \kappa_{i-1}\}$):*

1. *There is an $(n-1)$-cone function* Cone_{κ_0} *such that for ever* $0 \leq j \leq n-1$,

$$\mathrm{Rad}_j(\mathrm{Cone}_{\kappa_0}) \leq f(n),$$

where $f : \mathbb{R} \to \mathbb{R}$ *is a function depending only on* $\mathbf{C}$.

2. κ_i *is a full subcomplex of* κ *and for all vertices* $w_1, w_2 \in V_i$ *we have* $\{w_1, w_2\} \notin \kappa$, *for all* $0 \leq i \leq \ell$.

3. $\mathrm{lk}_{\kappa_i}(w) \cap \kappa_{i-1}$ *is either in* $\mathbf{C}$ *or can be written as the join of two elements from* $\mathbf{C}$ *for any* $w \in V_i, i \geq 1$.

Then for every $n \in \mathbb{N}_0$ *there exists a constant* $\mathcal{R}(n)$ *such that for every* $\kappa \in \mathbf{C}$ *with* $\dim \kappa = n$ *we have:* κ *has an* $(n-1)$*-cone function* Cone_κ *such that for all* $-1 \leq j \leq n-1$ *we have*

$$\mathrm{Rad}_j(\mathrm{Cone}_\kappa) \leq \mathcal{R}(n).$$

We can describe $\mathcal{R}(n)$ *recursively. We have* $\mathcal{R}(0) = 1$. *For* $n \geq 1$, *assume* $\mathcal{R}(k)$ *is known for* $k < n$. *Set*

$$S(n) = \max_{a,b \in \mathbb{N}: a+b+1=n} ((a+1)\mathcal{R}(a)+1)\mathcal{R}(b).$$

Then

$$\mathcal{R}(n) = S(n)^{\ell_n} f(n) + \sum_{j=1}^{\ell_n} S(n)^j.$$

Proof. We will prove the result by induction on the dimension n.

For $n = 0$, since $\kappa \neq \emptyset$ there exists $v \in \kappa(0)$ and thus we can define a (-1)-cone function via $\emptyset \mapsto \mathbb{1}_{[v]}$. Every cone function has (-1)-cone radius $= 1$, thus we take $\mathcal{R}(0) = 1$.

Fix $n \geq 1$ and $\kappa \in \mathbf{C}, \dim(\kappa) = n$ with filtration $\kappa_0 \subset \ldots \subset \kappa_\ell = \kappa$ as in the requirements. We will prove by induction on $0 \leq i \leq \ell$, that there is an $(n-1)$-cone function Cone_{κ_i} such that for every $-1 \leq k \leq n-1$

$$\mathrm{Rad}_k(\mathrm{Cone}_{\kappa_i}) \leq \mathcal{R}^{(i)}(n)$$

where $\mathcal{R}^{(i)}(n)$ is given inductively by

$$\mathcal{R}^{(0)}(n) = f(n), \qquad \mathcal{R}^{(i)}(n) = S(n)(\mathcal{R}^{(i-1)}(n) + 1) = S(n)^i f(n) + \sum_{j=1}^{i} S(n)^j.$$

For $i = 0$ this follows from Assumption 1. We proceed by induction on i. Fix $1 \leq i \leq \ell$ and assume there exists a constant $\mathcal{R}^{(i-1)}(n)$ and an $(n-1)$-cone function $\mathrm{Cone}_{\kappa_{i-1}}$ such that $\mathrm{Rad}_j(\mathrm{Cone}_{\kappa_{i-1}}) \leq \mathcal{R}^{(i-1)}(n)$ for all $-1 \leq j \leq n-1$. We want to apply Theorem 3.2 to the following setting:

$$X = \kappa, X' = \kappa_{i-1},\ \mathrm{Cone}_{X'} = \mathrm{Cone}_{\kappa_{i-1}},\ R'_j = \mathcal{R}^{(i-1)}(n),\ W = V_i = \kappa_i(0) \setminus \kappa_{i-1}(0).$$

We check that the conditions of the theorem are satisfied.

1. $W \cap X' = \emptyset$ by definition.

2. For every $w_1, w_2 \in W : \{w_1, w_2\} \notin \kappa_i \subset \kappa$ by assumption.

3. For every $w \in W, \operatorname{lk}_X(w) \cap X'$ is a non-empty simplicial complex, since by Assumption 3 we have $\operatorname{lk}_X(w) \cap X' \in \mathbf{C}$ or a join of two complexes in $\mathbf{C}$.

4. By Assumption 3, $\operatorname{lk}_X(w) \cap X'$ is either in $\mathbf{C}$ or is a join of two elements of $\mathbf{C}$. In the first case we use the induction hypothesis and get an $(n-2)$-cone function $\mathrm{Cone}_{\operatorname{lk}_X(w)\cap X'}$ such that
$$\mathrm{Rad}_j(\mathrm{Cone}_{\operatorname{lk}_X(w)\cap X'}) \leq \mathcal{R}(n-1).$$
In the second case, we can write $\operatorname{lk}_X(w) \cap X' = A * B$ for some $A, B \in \mathbf{C}$. Set $n_A = \dim A, n_B = \dim B$ then $n-1 \geq \dim(\operatorname{lk}_X(w) \cap X') = n_A + n_B + 1$, hence $n_A, n_B < n$. Thus by induction there exists an $(n_A - 1)$-cone function Cone_A with $\mathrm{Rad}_j(\mathrm{Cone}_A) \leq \mathcal{R}(n_A)$ for all $-1 \leq j \leq n_A - 1$ and similarly for B. Using Proposition 2.13 we get an $(n-1)$-cone function $\mathrm{Cone}_{\operatorname{lk}_X(w)\cap X'}$ such that
$$\mathrm{Rad}_j(\mathrm{Cone}_{\operatorname{lk}_X(w)\cap X'}) \leq ((n_A+1)\mathcal{R}(n_A)+1)\mathcal{R}(n_B) \leq S(n) \text{ for all } -1 \leq j \leq n-2.$$
Note that we simplified the bound coming from Proposition 2.13 by only considering the largest possible of the cases. We chose $S(n)$ to be the maximum over all the bounds that can appear at this step. Note that in particular $\mathcal{R}(n-1) \leq S(n)$. We set $R''_j = S(n)$.

Applying Theorem 3.2 we get an $(n-1)$-cone function $\mathrm{Cone}_{X' \cup W}$ such that
$$\mathrm{Rad}_j(\mathrm{Cone}_{X'\cup W}) \leq R''_{j-1}(R'_j + 1)$$
which in the original formulation means
$$\mathrm{Rad}_j(\mathrm{Cone}_{\kappa_i}) \leq S(n)(\mathcal{R}^{(i-1)}(n) + 1).$$
Thus we have $\mathcal{R}^{(i)}(n) = S(n)(\mathcal{R}^{(i-1)}(n) + 1)$ and solving the recursion gives the second expression of $\mathcal{R}^{(i)}(n)$.

Now, since $\kappa_\ell = \kappa$, we get an $(n-1)$-cone function Cone_κ with $\mathrm{Rad}(\mathrm{Cone}_\kappa) \leq \mathcal{R}^{(\ell)}(n)$. We would like to set $\mathcal{R}(n) = \mathcal{R}^{(\ell)}(n)$. But the bound has to hold for all possible $\kappa \in \mathbf{C}$ of dimension n. Since $\mathcal{R}^{(i)}(n)$ is increasing in i, we take $\mathcal{R}(n) = \mathcal{R}^{(\ell_n)}(n) = S(n)^{\ell_n} f(n) + \sum_{j=1}^{\ell_n} S(n)^j$. □

Remark 6.2. *Compared to [Abr96, Lemma 22] we changed condition 1. from just asking for κ_0 to be contractible to requiring an explicit cone function, and we removed the dependency on the underlying building, since the theorem also works in this wider generality. In particular, if we want to show that a certain class, for which Abramenko already showed that it satisfies the assumptions of [Abr96, Lemma 22], also satisfies the assumptions of this theorem, we need to check condition 1., while conditions 2. and 3. are usually already covered.*

6.2 Buildings of type A_n

To show that the complex opposite the fundamental chamber in a spherical building of type A_n has a cone function with bounded cone radius, we show that the class $\mathbf{C}_A$ from Definition 5.6 satisfies the assumptions of Theorem 6.1.

We will need some facts that appear in the proof of [Abr96, Proposition 12].

Fact 6.3. *If $T_{\mathcal{E}}(V) \in \mathbf{C}_A$, then there exists $\ell \in X_{\mathcal{E}}(V)$ such that ℓ is a line, i.e.,* $\dim(\ell) = 1$.

Let $\ell \in X_{\mathcal{E}}(V)$ be a line. Define

$$Y_0 = \{U \in T_{\mathcal{E}}(V) : \ell + U \in T_{\mathcal{E}}(V)\},$$

and further define κ_0 to be the subcomplex of $T_{\mathcal{E}}(V)$ spanned by Y_0.

Assume now that $\dim V = n+2$ and hence $\dim T_{\mathcal{E}}(V) = n$ (and $\operatorname{Flag} X$ is a building of type A_{n+1}). For $1 \leq i \leq n+1$, define $Y_i \subseteq X_{\mathcal{E}}(V)$ as follows:

$$Y_i = \{U \in X_{\mathcal{E}}(V) \mid \dim(U) \geq n+2-i \text{ or } U \in Y_0\}.$$

For every such i, let κ_i be the subcomplex of $T_{\mathcal{E}}(V)$ spanned by Y_i.

Fact 6.4. *We denote $Y = X_{\mathcal{E}}(V), \kappa = T_{\mathcal{E}}(V)$ and for every $U \in T_{\mathcal{E}}(V)$ we denote the link of U by $\operatorname{lk}_\kappa(U)$. For every $1 \leq i \leq n+1$ and every $U \in Y_i \setminus Y_{i-1}$, it holds that $\operatorname{lk}_\kappa(U) \cap \kappa_{i-1} = T_{\mathcal{E}'}(U) * T_{\overline{\mathcal{E}}}(V/U)$ such that*

- *The set $\mathcal{E}'$ is a set of subsets of U satisfying $\sum_{j=1}^{n+1} \binom{n}{j-1} e'_j \leq |K|$, hence $T_{\mathcal{E}'}(U) \in \mathbf{C}_A$.*
- *The set $\overline{\mathcal{E}}$ is a set of subsets of V/U that satisfying $\sum_{j=1}^{n+1} \binom{n}{j-1} \overline{e}_j \leq |K|$ and hence $T_{\mathcal{E}'}(V/U) \in \mathbf{C}_A$.*

Lemma 6.5. *There is an $(n-1)$-cone function $\operatorname{Cone}_{\kappa_0}$ such that for every $0 \leq k \leq n-1$,*

$$\operatorname{Rad}_k(\operatorname{Cone}_{\kappa_0}) \leq n+2.$$

Proof. Let Γ_0 be the subcomplex of κ_0 spanned by $Z_0 = \{U \in X_{\mathcal{E}}(V) : \ell \leq U\}$. For $1 \leq i \leq n+1$, we denote

$$Z_i = \{U \in Y_0 \mid \dim(U) \leq i \text{ or } U \in Z_0\}.$$

Further denote $\Gamma_i = \operatorname{Flag} Z_i$. Note that $\Gamma_{n+1} = \kappa_0$. We will show by induction that for every $0 \leq i \leq n+1$, there is an $(n-1)$-cone function $\operatorname{Cone}_{\Gamma_i}$ such that for every $-1 \leq k \leq n-1$, $\operatorname{Rad}_k(\operatorname{Cone}_{Y_i}) \leq i+1$.

For $i = 0$, we note that Γ_0 is the ball of radius 1 around ℓ in κ and can be written as $\Gamma_0 = \{\ell\} * \operatorname{lk}_\kappa(\ell)$. By Proposition 2.11, there is an $(n-1)$-cone function $\operatorname{Cone}_{\Gamma_0}$ such that for every $-1 \leq k \leq n-1$, $\operatorname{Rad}_k(\operatorname{Cone}_{\Gamma_0}) \leq 1$.

Let $1 \leq i \leq n+1$ and assume there is an $(n-1)$-cone function $\operatorname{Cone}_{\Gamma_{i-1}}$ such that for every $-1 \leq k \leq n-1$, $\operatorname{Rad}_k(\operatorname{Cone}_{\Gamma_{i-1}}) \leq i$. We will apply Theorem 3.2 in order to bound the cone radii of Γ_i. We note that any two $U_1, U_2 \in Y_i \setminus Y_{i-1}, U_1 \neq U_2$ are of the same dimension and thus are not connected by an edge. Let $U \in Z_i \setminus Z_{i-1}$. We will show that $\operatorname{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$ is a join of the vertex $\{U + \ell\}$ with the complex spanned by all the other vertices in $\operatorname{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$.

Note that $\operatorname{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$ is a clique complex, thus it is enough to show that for every vertex U' in $\operatorname{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}$ that is not $\ell + U$, there is an edge connecting U' and $\ell + U$. Let U' be such vertex.

First, we will deal with the case were $\dim(U') < i$. In that case (using the fact that $\dim(U) = i$), U' being in $\operatorname{lk}_{\kappa_0}(U)$ implies that $U' \leq U$ and thus $U' \leq U + \ell$, i.e., U' is connected by an edge to $U + \ell$ as needed.

Second, assume that $\dim(U') > i$. From the definition of Z_{i-1}, it follows that $U' \in Z_0$, i.e., that $\ell \leq U'$. Also, U' is in $\mathrm{lk}_{\kappa_0}(U)$ and thus $U \leq U'$. It follows that $U + \ell \leq U'$, i.e., U' is connected by an edge to $U + \ell$ as needed.

By Proposition 2.11, for every $U \in Z_i \setminus Z_{i-1}$, there is an $(n-2)$-cone function $\mathrm{Cone}_{\mathrm{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}}$ such that for every $-1 \leq k \leq n-2$, $\mathrm{Rad}_k(\mathrm{Cone}_{\mathrm{lk}_{\kappa_0}(U) \cap \Gamma_{i-1}}) \leq 1$.

By Theorem 3.2 (with $R''_k = 1$ for every k), it follows that there is an $(n-1)$-cone function Cone_{Γ_i} such that for every $-1 \leq k \leq n-1$, $\mathrm{Rad}_k(\mathrm{Cone}_{\Gamma_i}) \leq i+1$ as needed. □

Corollary 6.6. *For every $n \in \mathbb{N} \cup \{0\}$ there is a constant $\mathcal{R}(n)$ such that for every $T_{\mathcal{E}}(V) \in \mathbf{C}_A$ with $\dim T_{\mathcal{E}}(V) = n$ there in an $(n-1)$-cone function $\mathrm{Cone}_{T_{\mathcal{E}}(V)}$ such that*

$$\mathrm{Rad}_k(\mathrm{Cone}_{T_{\mathcal{E}}(V)}) \leq \mathcal{R}(n), \forall -1 \leq k \leq n-1.$$

In particular, if Δ is a building of type A_n over a field K with $|K| \geq 2^{n-1}$ and $a \in \Delta$ a simplex, then there exists an $(n-2)$-cone function $\mathrm{Cone}_{\Delta^0(a)}$ of $\Delta^0(a)$ such that

$$\mathrm{Rad}_j(\mathrm{Cone}_{\Delta^0(a)}) \leq \mathcal{R}(n-1), \forall -1 \leq j \leq n-2.$$

Proof. The result follows from Theorem 6.1 together with Lemma 6.5 and Fact 6.4. In the notation of Theorem 6.1, we have $f(n) = n+2$ and $\ell_n = n+1$. □

6.3 Buildings of type C_n

The class $\mathbf{C}$ that will cover the case of opposition complexes in buildings of type C_n will be the union of class $\mathbf{C}_A$ (see Definition 5.6) and $\mathbf{C}_C$ (see Definition 5.10).

Proposition 6.7. *The class $\mathbf{C}$ satisfies the requirements of Theorem 6.1.*

Corollary 6.8. *Let $\Delta = \mathrm{Flag}\, X(V)$ be a classical C_n building as described in Section 5.3.3 (note that $\dim \Delta = n-1$). Assume that K is infinite or $K = \mathbb{F}_q$ and*

- *$q \geq 2^{2n-2}$ if f is alternating and $Q = 0$,*
- *$q \geq 2^{4n-4}$ if $m = 2n$ and $\sigma \neq \mathrm{id}$,*
- *$q \geq 2^{2n-1}$ if $m = 2n+1$ or $2n+2$.*

In particular, if Δ is the building coming from the BN-pair of a Chevalley group of type B_n or C_n over the field $\mathbb{F}_q$ we require $q \geq 2^{2n-1}$.

Then for any $a \in \Delta$ there exists an $(n-2)$-cone function $\mathrm{Cone}_{\Delta^0(a)}$ of $\Delta^0(a)$ with

$$\mathrm{Rad}_j(\mathrm{Cone}_{\Delta^0(a)}) \leq \mathcal{R}(n-1), \qquad -1 \leq j \leq n-2$$

where $\mathcal{R}(n-1)$ is as in Theorem 6.1 with $f(n) = 2n+1$ and $\ell_n = 2n+2$.

The main part of the proof of Proposition 6.7 is covered by the following lemma.

Lemma 6.9. *Let $(T_{\mathcal{E}}(V), \mathrm{Flag}\, X) \in \mathbf{C}_C$. Set $n = \dim \mathrm{Flag}\, X \geq 1$. By [Abr96], there exists a 1-dimensional subspace $\ell \in X_{\mathcal{E}}(V)$. Define $X_0 = \{U \in X \mid U$ satisfies $1, 2$ or $3\}$ where*

1. *$\ell \leq U$ and $U \pitchfork \mathcal{E}$;*

2. $\ell \nleq U \leq \ell^\perp$ *and* $U \pitchfork \mathcal{E} \cup (\mathcal{E} + \ell)$*;*

3. $U \nleq \ell^\perp, \dim U > 1$ *and* $U \pitchfork \mathcal{E} \cup (\mathcal{E} \cap \ell^\perp) \cup (\mathcal{E} + \ell) \cup ((\mathcal{E} \cap \ell^\perp) + \ell)$*.*

Then $\kappa_0 := \operatorname{Flag} X_0$ *has an* $(n-1)$*-cone function* $\operatorname{Cone}_{\kappa_0}$ *such that*

$$\operatorname{Rad}_j(\operatorname{Cone}_{\kappa_0}) \leq 2n+3 \qquad \textit{for all } -1 \leq j \leq n-1.$$

Proof. Let $(T_\mathcal{E}(V), \operatorname{Flag} X) \in \mathbf{C}_C$ as above. Thus $\dim \operatorname{Flag} X = n$ and $\operatorname{Flag} X$ is of type C_{n+1}. In particular, any totally isotropic subspace of V has dimension at most $n+1$.

We will apply Theorem 3.2 inductively to two different filtrations of κ_0. The first one is defined as follows.

$$\begin{aligned} &Y_0 = \{U \in X \text{ satisfying 1.}\} \\ &Y_i = Y_{i-1} \cup \{U \in X \mid \dim U = i, U \text{ satisfies 2.}\} = \{U \in X \mid \dim U \leq i, U \text{ satisfies 2.}\} \cup Y_0 \\ &\quad \text{for } 1 \leq i \leq n+1. \end{aligned}$$

We set $\Gamma_i = \operatorname{Flag} Y_i$.

The second filtration is given by

$$\begin{aligned} Z_0 &= Y_{n+1} \\ Z_i &= Z_{i-1} \cup \{U \in X \mid \dim U = n+1-(i-1), U \text{ satisfies 3.}\} \\ &= Z_0 \cup \{U \in X \mid \dim U \geq n+1-(i-1), U \text{ satisfies 3.}\} \end{aligned}$$

for $1 \leq i \leq n+1$. We set $\zeta_i = \operatorname{Flag} Z_i$. We have that $\Gamma_{n+1} = \zeta_0$ and $\zeta_{n+1} = \kappa_0$.

Next, we want to show that for each Γ_i there exists an $(n-1)$-cone function $\operatorname{Cone}_{\Gamma_i}$ such that $\operatorname{Rad}_j(\operatorname{Cone}_{\Gamma_i}) \leq i+1$ for all $-1 \leq j \leq n-1, 0 \leq i \leq n+1$. We do this by induction on i.

For $i = 0$ we have $\Gamma_0 = \operatorname{st}_\kappa(\ell) = \{\ell\} * \operatorname{lk}_\kappa(\ell)$. Hence by Proposition 2.11 there exists a cone function $\operatorname{Cone}_{\Gamma_0}$ with $\operatorname{Rad}_j(\operatorname{Cone}_{\Gamma_0}) \leq 1$ for all $-1 \leq j \leq n-1$.

For the following step, fix $1 \leq i \leq n+1$. We want to apply Theorem 3.2 to the following set-up. Here κ_0 corresponds to the complex called X in Theorem 3.2, Γ_{i-1} corresponds to X' , and $Y_i \setminus Y_{i-1} = W_i$. We check that the conditions of Theorem 3.2 are satisfied.

For condition 1. note that $W_i \cap \Gamma_{i-1} = \emptyset$ by definition. Condition 2. holds since all subspaces in W_i have the same dimension, hence they are not connected in Δ and thus not in κ.

For 3. and 4. let $w \in W_i$. Then $\operatorname{lk}_{\kappa_0}(w) = \operatorname{Flag}\{u \in \kappa_0(0) \mid u < w \text{ or } w > u\}$. Furthermore, for $u \in Y_{i-1}$ we have $u + \ell \in X_0$, since $u \pitchfork \mathcal{E} + \ell \iff u + \ell \pitchfork \mathcal{E}$, see [Abr96]. Since $\ell \leq w + \ell$, we have $w + \ell \in Y_0 \subset Y_{i-1}$. Next, we want to show that $\operatorname{lk}_{\kappa_0}(w) \cap \Gamma_{i-1}$ is a join of $\{w+\ell\}$ with the flag complex of all other vertices from $\operatorname{lk}_{\kappa_0}(w) \cap \Gamma_{i-1}$. Since $\operatorname{lk}_{\kappa_0}(w) \cap \Gamma_{i-1}$ is a clique complex, it suffices to check that every vertex different from $w + \ell$ is adjacent to $w + \ell$. Let $u \in (\operatorname{lk}_{\kappa_0}(w) \cap \Gamma_{i-1})(0) \setminus \{w + \ell\}$. Then we differentiate two cases:

- Case 1: $\dim u \leq i-1$: then $u \leq w \leq w + \ell$
- Case 2: $\dim u > i$ then $u \in Y_0$, thus $w \leq u$ and $\ell \leq u$ thus $w + \ell \leq u$.

Hence any other vertex is connected to $w + \ell$ by an edge and thus

$$\Gamma_0 = \{w + \ell\} * \operatorname{Flag}\left((\operatorname{lk}_{\kappa_0}(w) \cap \Gamma_{i-1})(0) \setminus \{w + \ell\}\right).$$

By Proposition 2.11 we have $\mathrm{Rad}_k(\mathrm{Cone}_{\mathrm{lk}_{\kappa_0}(w)\cap\Gamma_{i-1}}) \leq 1$.

Thus we can apply Theorem 3.2 and get an $n-1$-done function Cone_{Γ_i} satisfying $\mathrm{Rad}_j(\mathrm{Cone}_{\Gamma_i}) \leq i+1$.

We treat the second filtration similarly. We want to show that for every $0 \leq i \leq n+1$ there is an $(n-1)$-cone function Cone_{ζ_i} such that $\mathrm{Rad}_j(\mathrm{Cone}_{\zeta_i}) \leq n+2+i$. We again proceed by induction on i.

For $i=0$ we have that $\zeta_0 = \Gamma_{n+1}$ has cone function with radius $\leq n+2$ by the above reasoning.

Now fix $1 \leq i \leq n+1$. We want to use Theorem 3.2 with κ_0 corresponding to X, ζ_{i-1} to X', and $Z_i \setminus Z_{i-1} = W$. Then conditions 1. and 2. of Theorem 3.2 are satisfied by the same argument as above.

For conditions 3. and 4., fix $w \in W$ and consider $\mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1} = \mathrm{Flag}\,\{u \in Z_{i-1} \mid u < w \text{ or } u > w\}$. We show that every vertex (different from $w \cap \ell^\perp$) is connected to $w \cap \ell^\perp$ hence $\mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1} = \{w \cap \ell^\perp\} * \mathrm{Flag}\,\{u \in Z_{i-1} \setminus \{w \cap \ell^\perp\} \mid u < w \text{ or } u > w\}$. Thus by Proposition 2.11, $\mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}$ has a cone radius bounded by 1. Applying Theorem 3.2 we get a cone function Cone_{ζ_i} satisfying

$$\mathrm{Rad}_k(\mathrm{Cone}_{\zeta_i}) \leq \mathrm{Rad}_k(\mathrm{Cone}_{\zeta_{i-1}}) + 1 \leq n+2+i.$$

To show that every vertex (different from $w \cap \ell^\perp$) is connected to $w \cap \ell^\perp$, let $u \in \mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}(0) = \{u \in Z_{i-1} \mid u < w \text{ or } u > w\}$. If $w \leq u$ then $w \cap \ell^\perp \leq u$ and thus $\{u, w \cap \ell^\perp\} \in \mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}$. If $u \leq w$, then $\dim u < \dim w = n+1-i+1$ hence $u \in Z_0$. Therefore u satisfies either 1 or 2. But if it satisfies 1, then $\ell \leq u \leq w$, a contradiction to $w \in Z_i \setminus Z_{i-1}$. Hence u satisfies 2, and in particular $u \leq \ell^\perp$. Thus $u \leq w \cap \ell^\perp$ and hence $\{u, w \cap \ell^\perp\} \in \mathrm{lk}_{\kappa_0}(w) \cap \zeta_{i-1}$.

Since $\kappa_0 = \zeta_{n+1}$ we get the desired result. □

Proof of Proposition 6.7. We proof that every $(\kappa, \Delta) \in \mathbf{C}$ has a filtration satisfying the assumptions of Theorem 6.1 by induction on $n = \dim \Delta$.

Let $n = 0$. Hence Δ will be a building of type A_1 or C_1. By [Abr96] there exists at least on 1-dimensional subspace $\ell \in X_{\mathcal{E}}(V)$ and thus κ will be non-empty.

For the inductive step, fix $n \geq 1$ and set $f(n) = 2n+3$. We distinguish between two cases:

If $(\kappa, \Delta) \in \mathbf{C}_A$, Section 6.2 gives the desired filtration with $f_A(n) = n+2 \leq f(n)$ and length of the filtration being $n+1$.

If $(\kappa, \Delta) \in \mathbf{C}_C$, we can use the notation as in Definition 5.10, e.g. $(\kappa, \Delta) = (T_{\mathcal{E}}(V), \mathrm{Flag}\,X)$. As in Lemma 6.9 we fix a 1-dimensional subspace $\ell \in X_{\mathcal{E}}(V)$ and define $X_0 = \{U \in X \mid U \text{ satisfies } 1, 2 \text{ or } 3\}$ (where 1,2,3 are as in the lemma) and $\kappa_0 = \mathrm{Flag}\,X_0$. Hence, by Lemma 6.9, there exists an $(n-1)$-cone function Cone_{κ_0} such that

$$\mathrm{Rad}_j(\mathrm{Cone}_{\kappa_0}) \leq 2n+3 = f(n) \text{ for all } -1 \leq j \leq n-1.$$

Next, we define the desired filtration in two steps. Set $Z = \{U \in X_{\mathcal{E}}(V) \mid \ell \leq U \text{ or } U \pitchfork \mathcal{E} + \ell\}$ and $Y_0 = X_0$. Note that $Y_0 \subseteq Z$. For the first half of the filtration we define

$$Y_i = \{U \in Z \mid U \in Y_0 \text{ or } \dim U \leq i\}, \kappa_i = \mathrm{Flag}\,Y_i,\ 1 \leq i \leq n+1.$$

For the second half we set

$$Y_i = \{U \in X_{\mathcal{E}}(V) \mid U \in Z \text{ or } \dim U \geq 2n+3-i\}, \kappa_i = \mathrm{Flag}\,Y_i,\ n \leq i \leq 2n+2.$$

Then $\kappa_{2n+2} = \kappa$ and [Abr96, Propostition 13] shows that the filtration satisfies assumptions 2 and 3.

□

6.4 Buildings of type D_n

The proof that opposition complexes of buildings of type D_n are coboundary expanders is the most involved case. We use the notation introduced in Section 5.3.4.

The following proposition can be seen as a quantitative version of [Abr96, Corollary 17 (ii)].

Proposition 6.10. *Let $\tau = \{E_1 < \cdots < E_r\} \in \tilde{\Delta}$ and set $\mathcal{E}(\tau) = \{E_i, E_i^{\perp} \mid 1 \leq i \leq r\}$. If $T_{\mathcal{E}(\tau)}(V)$ has an $(n-1)$-cone function $\mathrm{Cone}_{T_{\mathcal{E}(\tau)}(V)}$ with $\mathrm{Rad}(\mathrm{Cone}_{T_{\mathcal{E}(\tau)}(V)}) \leq c$ for some $c \geq 0$ then there exists an $(n-1)$-cone function $\mathrm{Cone}_{\tilde{T}_{\mathcal{E}(\tau)}(V)}$ with $\mathrm{Rad}(\mathrm{Cone}_{\tilde{T}_{\mathcal{E}(\tau)}(V)}) \leq 2c$.*

Proof. For shorter notation, we write $T = T_{\mathcal{E}(\tau)}(V), Y = X_{\mathcal{E}(\tau)}(V)$ and $\tilde{T} = \tilde{T}_{\mathcal{E}(\tau)}(V), \tilde{Y} = \tilde{X}_{\mathcal{E}(\tau)}(V)$.

The idea of the proof is to construct an $(n-1)$-cone function for $\tilde{T}$ given a cone function

$$\mathrm{Cone}_T : \oplus_{j=-1}^{n-1} C_j(T) \to \oplus_{j=0}^{n} C_j(T)$$

by defining maps $f_j : C_j(\tilde{T}) \to C_j(T)$ for $-1 \leq j \leq n-1$ and $g_j : C_j(T) \to C_j(\tilde{T})$ for $0 \leq j \leq n$ such that $g_{j+1} \circ \mathrm{Cone}_T|_{C_j(T)} \circ f_j$ gives rise to a cone function of $\tilde{T}$.

As already observed in [Abr96], T is a subdivision of $\tilde{T}$ in the following way. Note that $\tilde{Y} \subset Y$ and $Y \setminus \tilde{Y} = \{U \in Y \mid \dim U = n-1\}$. If two subspaces $U, W \in \tilde{Y}$ are connected by an edge in T then the same is true in $\tilde{T}$, but we have additional edges in $\tilde{T}$, namely in the case when $\dim U = \dim W = n$ and $\dim(U \cap W) = n-1$. To go from $\tilde{T}$ to T, we subdivide the edge $\{U, W\}$ by adding the vertex $U \cap W$ and connecting U, W and every vertex $U' \in \tilde{Y}$ for which $\{U', U, W\} \in \tilde{T}$ to $U \cap W$.

Recall from the discussion above that there are two different types of n-dimensional spaces in $\tilde{X}$ and that spaces from the same type cannot have an intersection of dimension $n-1$. Furthermore, each $U \in X$ of dimension $n-1$ is contained in exactly two totally isotropic spaces of dimension n. If $U \in Y$ of dimension $n-1$, i.e. $U \tilde{\pitchfork} \mathcal{E}$, with $W_1, W_2 \in X$ of dimension n such that $U = W_1 \cap W_2$ then by [Abr96, Lemma 31 (iii)] we have that $W_i \tilde{\pitchfork} \mathcal{E}$ and hence $W_1, W_2 \in Y$.

In particular, any simplex in $\tilde{T}$ contains at most two vertices which are spaces of dimension n.

For each $U \in Y \setminus \tilde{Y}$ pick one of the two n-dimensional subspaces containing it and denote it by W_U. Note that if $W \in Y$ is connected to U by an edge in T, i.e. $W \subseteq U$ or $U \subseteq W$ then $\{W_U, W\} \in T(1)$ if and only if $W \neq W_U$.

We define the following two subsets of $T(j)$ for $0 \leq j \leq n$:

$$I_j = \{\sigma \in T(j) \mid \exists U \in \sigma(0) : \dim U = n-1\},\ J_j = \{\sigma \in T(j) \mid \exists U \in \sigma(0) : \dim U = n-1 \text{ and } W_U \notin \sigma(0)\}.$$

Let $\sigma \in J_j$ then we define $\sigma\langle U, W_U \rangle \in \overrightarrow{T(j)}$ to be the oriented simplex with vertex W_U at the position of U instead of U. This is well defined by the above observation. Similarly, if $\sigma \in \overrightarrow{\tilde{T}}$ with $W_1, W_2 \in \sigma(0), \dim W_1 = \dim W_2 = n$ and $U = W_1 \cap W_2$ has dimension $n-1$, we write $\sigma\langle W_i, U \rangle$ for the oriented simplex where we replaced W_i with U. This is now a well-defined simplex in $\overrightarrow{T(j)}$.

Next, we define the function $f_j : C_j(\tilde{T}) \to C_j(T), 0 \leq j \leq n-1$ by its action on chains of the form $\mathbb{1}_\sigma$ and then extending $\mathbb{Z}$-linearly:

$$f_j(\mathbb{1}_\sigma) = \begin{cases} \mathbb{1}_\sigma & \text{if } \sigma \in \overrightarrow{T(j)} \cap \overrightarrow{\tilde{T}(j)} \\ \mathbb{1}_{\sigma\langle W_1, U \rangle} + \mathbb{1}_{\sigma\langle W_2, U \rangle} & \text{if } \sigma \in \overrightarrow{\tilde{T}(j)} \setminus \overrightarrow{T(j)}, W_1, W_2 \in \sigma(0), \dim W_i = n, U = W_1 \cap W_2, \dim U = n-1. \end{cases}$$

On the other hand, we define

$$g_j : C_j(T) \to C_j(\tilde{T})$$
$$\mathbb{1}_\sigma \mapsto \begin{cases} \mathbb{1}_\sigma & \text{if } \sigma \in \overrightarrow{T(j)} \cap \overrightarrow{\tilde{T}(j)} \\ \mathbb{1}_{\sigma\langle U, W_U\rangle} & \text{if } \sigma \in J_j \\ 0 & \text{if } \sigma \in I_j \setminus J_j. \end{cases}$$

We define $\mathrm{Cone}^j_{\tilde{T}} = g_{j+1} \circ \mathrm{Cone}^j_T \circ f_j, -1 \leq j \leq n-1$, where $\mathrm{Cone}^j_T := \mathrm{Cone}_T|_{C_j(T)}$ and $\mathrm{Cone}_{\tilde{T}} = \oplus_{j=-1}^{n-1} \mathrm{Cone}^j_{\tilde{T}}$. The majority of the rest of the proof is dedicated to showing that this is indeed a cone function. As concatenation of $\mathbb{Z}$-linear functions it is $\mathbb{Z}$-linear. If $[U]$ is the appex of Cone_T then the appex of $\mathrm{Cone}_{\tilde{T}}$ is again $[U]$ if $\dim U \neq n-1$ and otherwise it is $[W_U]$.

Hence, we are left with showing the cone equation. Let $\sigma \in \overrightarrow{T(j)}$. We write $\mathrm{Cone}_T(\mathbb{1}_\sigma) = \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau \mathbb{1}_\tau$ for $\lambda_\tau \in \mathbb{Z}$. In the following, if $\tau \in I_{j+1}$ is a simplex, then U will always denote the subspace of dimension $n-1$ in $\tau(0)$. Additionally, $[\sigma : \tau]$ will denote the oriented incidence number, which is 0 if $\tau \not\subseteq \sigma$ and 1 or -1 if $\tau \subset \sigma$ depending on the position of the vertex of σ that was removed to obtain τ. It is chosen in such a way that for $\sigma \in \overrightarrow{T(j+1)}$ we have $\partial_{j+1} \mathbb{1}_\sigma = \sum_{\tau \in \overrightarrow{T(j)}} [\sigma : \tau] \mathbb{1}_\tau$. Then

$$\begin{aligned} \partial_{j+1}(g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_\sigma))) &= \partial_{j+1}(\mathrm{Cone}_T(\mathbb{1}_\sigma)) - \sum_{\tau \in I_{j+1}} \lambda_\tau \partial_{j+1} \mathbb{1}_\tau + \sum_{\tau \in J_{j+1}} \lambda_\tau \partial_{j+1} \mathbb{1}_{\tau\langle U, W_U\rangle} \\ &= \mathbb{1}_\sigma - \mathrm{Cone}_T(\partial_j \mathbb{1}_\sigma) - \sum_{\tau \in I_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)}} [\tau : \gamma] \mathbb{1}_\gamma \\ &\quad + \sum_{\tau \in J_{j+1}} \lambda_\tau [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}} + \sum_{\tau \in J_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_{\gamma\langle U, W_U\rangle} \end{aligned}$$

where we used that

$$\partial_{j+1} \mathbb{1}_{\tau\langle U, W_U\rangle} = \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_{\gamma\langle U, W_U\rangle} + [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}}.$$

Note that $(\tau\langle U, W_U\rangle) \setminus \{W_U\} = \tau \setminus \{U\}$ and $[\tau : \gamma\langle U, W_U\rangle] = [\tau : \gamma]$.

We take a closer look at the second half of that expression:

$$\begin{aligned} &- \sum_{\tau \in I_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)}} [\tau : \gamma] \mathbb{1}_\gamma + \sum_{\tau \in J_{j+1}} \lambda_\tau [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}} + \sum_{\tau \in J_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_{\gamma\langle U, W_U\rangle} \\ =& - \sum_{\tau \in I_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_\gamma + \sum_{\tau \in J_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_{\gamma\langle U, W_U\rangle} \\ &- \sum_{\tau \in I_{j+1}} \lambda_\tau [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}} + \sum_{\tau \in J_{j+1}} \lambda_\tau [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}} \\ =& - \sum_{\tau \in I_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_\gamma + \sum_{\tau \in J_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_{\gamma\langle U, W_U\rangle} \\ &- \sum_{\tau \in I_{j+1} \setminus J_{j+1}} \lambda_\tau [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}} \end{aligned}$$

On the other hand, let $\tilde{\lambda}_\gamma \in \mathbb{Z}$ such that $\operatorname{Cone}_T(\partial \mathbb{1}_\sigma) = \sum_{\gamma \in \overrightarrow{T(j)}} \tilde{\lambda}_\gamma \mathbb{1}_\gamma$. Then

$$g_j(\operatorname{Cone}_T(\partial \mathbb{1}_\sigma)) = \operatorname{Cone}_T(\partial \mathbb{1}_\sigma) - \sum_{\gamma \in I_j} \tilde{\lambda}_\gamma \mathbb{1}_\gamma + \sum_{\gamma \in J_j} \tilde{\lambda}_\gamma \mathbb{1}_{\gamma\langle U, W_U \rangle}.$$

What are the $\tilde{\lambda}_\gamma$ in terms of the λ_τ?

$$\begin{aligned}
\operatorname{Cone}_T(\partial \mathbb{1}_\sigma) = \mathbb{1}_\sigma - \partial_{j+1} \operatorname{Cone}_T(\mathbb{1}_\sigma) &= \mathbb{1}_\sigma - \partial_{j+1} \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau \mathbb{1}_\tau \\
&= \mathbb{1}_\sigma - \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)}} [\tau : \gamma] \mathbb{1}_\gamma \\
&= \mathbb{1}_\sigma - \sum_{\gamma \in \overrightarrow{T(j)}} \left(\sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_\gamma.
\end{aligned}$$

Hence if $\gamma \neq \sigma$ then

$$\tilde{\lambda}_\gamma = - \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma]$$

and otherwise

$$\tilde{\lambda}_\sigma = 1 - \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \sigma].$$

To compare the expression of $g_j(\operatorname{Cone}_T(\partial_j \mathbb{1}_\sigma))$ with $\partial_{j+1}(g_{j+1}(\operatorname{Cone}_T(\mathbb{1}_\sigma)))$ we furthermore need the following observation comparing the sets I_j with I_{j+1} and J_j with J_{j+1}.

$$\left\{ (\tau, \gamma) : \gamma \in I_j, \tau \in \overrightarrow{T(j+1)}, \gamma \subset \tau \right\} = \left\{ (\tau, \gamma) : \tau \in I_{j+1}, \gamma \in \overrightarrow{T(j)}, \gamma \subset \tau, \gamma \neq \tau \setminus \{U\} \right\}$$

and

$$\begin{aligned}
&\left\{ (\tau, \gamma) : \gamma \in J_j, \tau \in \overrightarrow{T(j+1)}, \gamma \subset \tau \right\} \\
=&\left\{ (\tau, \gamma) : \tau \in J_{j+1}, \gamma \in \overrightarrow{T(j)}, \gamma \subset \tau, \gamma \neq \tau \setminus \{U\} \right\} \cup \{ (\tau, \gamma) : \tau \in I_{j+1} \setminus J_{j+1}, \gamma = \tau \setminus \{W_U\} \}.
\end{aligned}$$

We use the above result to further investigate the second half of the expression for $g_j(\operatorname{Cone}_T(\partial \mathbb{1}_\sigma))$. We first assume that $\sigma \notin I_j$.

$$\begin{aligned}
&- \sum_{\gamma \in I_j} \tilde{\lambda}_\gamma \mathbb{1}_\gamma + \sum_{\gamma \in J_j} \tilde{\lambda}_\gamma \mathbb{1}_{\gamma\langle U, W_U \rangle} = - \sum_{\gamma \in I_j} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_\gamma + \sum_{\gamma \in J_j} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_{\gamma\langle U, W_U \rangle} \\
&= \sum_{\tau \in I_{j+1}} \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \lambda_\tau \mathbb{1}_\gamma - \sum_{\tau \in J_{j+1}} \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \lambda_\tau \mathbb{1}_{\gamma\langle U, W_U \rangle} \\
&- \sum_{\tau \in I_{j+1} \setminus J_{j+1}} [\tau : \tau \setminus \{W_U\}] \lambda_\tau \mathbb{1}_{(\tau \setminus \{W_U\})\langle U, W_U \rangle}.
\end{aligned}$$

As second case, assume $\sigma \in I_j \setminus J_j$.

$$g_j(\mathrm{Cone}_T(\partial \mathbb{1}_\sigma)) = \mathrm{Cone}_T(\partial \mathbb{1}_\sigma) - \sum_{\gamma \in I_j \setminus \{\sigma\}} \tilde{\lambda}_\gamma \mathbb{1}_\gamma + \sum_{\gamma \in J_j} \tilde{\lambda}_\gamma \mathbb{1}_{\gamma\langle U, W_U\rangle} - \tilde{\lambda}_\sigma \mathbb{1}_\sigma$$

$$= \mathrm{Cone}_T(\partial \mathbb{1}_\sigma) - \sum_{\gamma \in I_j \setminus \{\sigma\}} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_\gamma + \sum_{\gamma \in J_j} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_{\gamma\langle U, W_U\rangle}$$

$$- \left(1 - \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \sigma] \right) \mathbb{1}_\sigma$$

$$= \mathrm{Cone}_T(\partial \mathbb{1}_\sigma) - \sum_{\gamma \in I_j} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_\gamma + \sum_{\gamma \in J_j} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_{\gamma\langle U, W_U\rangle} - \mathbb{1}_\sigma$$

As third and last case, we consider $\sigma \in J_j \subset I_j$.

$$g_j(\mathrm{Cone}_T(\partial \mathbb{1}_\sigma)) = \mathrm{Cone}_T(\partial \mathbb{1}_\sigma) - \sum_{\gamma \in I_j \setminus \{\sigma\}} \tilde{\lambda}_\gamma \mathbb{1}_\gamma + \sum_{\gamma \in J_j \setminus \{\sigma\}} \tilde{\lambda}_\gamma \mathbb{1}_{\gamma\langle U, W_U\rangle} - \tilde{\lambda}_\sigma \mathbb{1}_\sigma + \tilde{\lambda}_\sigma \mathbb{1}_{\sigma\langle U, W_U\rangle}$$

$$= \mathrm{Cone}_T(\partial \mathbb{1}_\sigma) - \sum_{\gamma \in I_j \setminus \{\sigma\}} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_\gamma + \sum_{\gamma \in J_j \setminus \{\sigma\}} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_{\gamma\langle U, W_U\rangle}$$

$$- \left(1 - \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \sigma] \right) \mathbb{1}_\sigma + \left(1 - \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \sigma] \right) \mathbb{1}_{\sigma\langle U, W_U\rangle}$$

$$= \mathrm{Cone}_T(\partial \mathbb{1}_\sigma) - \sum_{\gamma \in I_j} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_\gamma + \sum_{\gamma \in J_j} \left(- \sum_{\tau \in \overrightarrow{T(j+1)}} \lambda_\tau [\tau : \gamma] \right) \mathbb{1}_{\gamma\langle U, W_U\rangle} - \mathbb{1}_\sigma + \mathbb{1}_{\sigma\langle U, W_U\rangle}$$

We now look at almost the complete cone equation for $\mathrm{Cone}_{\tilde{T}}$, still not considering f_j. Let $\sigma \in \overrightarrow{T(j)} \cap \overrightarrow{\tilde{T}(j)}$, then we get

$$\partial_{j+1} g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_\sigma)) + g_{j+1}(\mathrm{Cone}_T(\partial_j \mathbb{1}_\sigma)) = \mathbb{1}_\sigma - \mathrm{Cone}_T(\partial_j \mathbb{1}_\sigma)$$
$$- \sum_{\tau \in I_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_\gamma + \sum_{\tau \in J_{j+1}} \lambda_\tau \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \mathbb{1}_{\gamma\langle U, W_U\rangle}$$
$$- \sum_{\tau \in I_{j+1} \setminus J_{j+1}} \lambda_\tau [\tau : \tau \setminus \{U\}] \mathbb{1}_{\tau \setminus \{U\}}$$
$$+ \mathrm{Cone}_T(\partial \mathbb{1}_\sigma) + \sum_{\tau \in I_{j+1}} \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \lambda_\tau \mathbb{1}_\gamma - \sum_{\tau \in J_{j+1}} \sum_{\gamma \in \overrightarrow{T(j)} : \gamma \neq \tau \setminus \{U\}} [\tau : \gamma] \lambda_\tau \mathbb{1}_{\gamma\langle U, W_U\rangle}$$
$$- \sum_{\tau \in I_{j+1} \setminus J_{j+1}} [\tau : \tau \setminus \{W_U\}] \lambda_\tau \mathbb{1}_{(\tau \setminus \{W_U\})\langle U, W_U\rangle}.$$

The orange, teal and blue parts cancel each other out. It remains to show that the same is true for the purple parts. Note that the sums in the two purple parts run over the same elements. Hence we need to show that for any $\tau \in I_{j+1} \setminus J_{j+1}$ we have

$$[\tau : \tau \setminus \{U\}]\mathbb{1}_{\tau\setminus\{U\}} = -[\tau : \tau \setminus \{W_U\}]\mathbb{1}_{(\tau\setminus\{W_U\})\langle U,W_U\rangle}$$

Note that as unoriented simplices $\tau \setminus \{U\}$ is the same as $(\tau \setminus \{W_U\})\langle U, W_U\rangle$ (in both cases we have all the vertices of τ except U). Let $\tau = [U_0, \ldots, U_{j+1}]$ and assume that $U = U_\ell$ and $W_U = U_k$ for some $0 \le \ell, k \le j+1$. Then $[\tau : \tau \setminus \{U\}] = (-1)^\ell$ and $[\tau : \tau \setminus \{W_U\}] = (-1)^k$. Further note that we can obtain$(\tau \setminus \{W_U\})\langle U, W_U\rangle$ from $\tau \setminus \{U\}$ by cyclically permuting the vertices of index between $\min\{\ell, k\}$ and $\max\{\ell, k\}-1$. The cycle has length $|k-\ell|-1$ and thus signature $(-1)^{k-\ell-1}$. Hence $(-1)^k\mathbb{1}_{(\tau\setminus\{W_U\})\langle U,W_U\rangle} = (-1)^k \cdot (-1)^{k-\ell-1}\mathbb{1}_{\tau\setminus\{U\}} = -(-1)^\ell\mathbb{1}_{\tau\setminus\{U\}}$ which is what we wanted to show.

To summarize, we get

$$\partial_{j+1}g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_\sigma) + g_{j+1}(\mathrm{Cone}_T(\partial_j\mathbb{1}_\sigma)) = \mathbb{1}_\sigma.$$

Analogously, we get for $\sigma \in I_j \setminus J_j$

$$\partial_{j+1}g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_\sigma) + g_{j+1}(\mathrm{Cone}_T(\partial_j\mathbb{1}_\sigma)) = 0$$

and for $\sigma \in J_j$

$$\partial_{j+1}g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_\sigma) + g_{j+1}(\mathrm{Cone}_T(\partial_j\mathbb{1}_\sigma)) = \mathbb{1}_{\sigma\langle U,W_U\rangle}.$$

Note that if $\sigma \in \overrightarrow{T(j)} \cap \overrightarrow{\tilde{T}(j)}$, then $f_j(\mathbb{1}_\sigma) = \mathbb{1}_\sigma$ which shows that the cone equation holds in this case. To conclude the proof, we need to check to cone equation also for the case where $\sigma \in \overrightarrow{\tilde{T}(j)} \setminus \overrightarrow{T(j)}$. In this case $\sigma(0)$ contains two subspaces W_1, W_2 with $\dim W_i = n$ and $U := W_1 \cap W_2$ has dimension $n-1$. Without loss of generality, assume that $W_U = W_1$. We get $f_j(\mathbb{1}_\sigma) = \mathbb{1}_{\sigma\langle W_1,U\rangle} + \mathbb{1}_{\sigma\langle W_2,U\rangle}$. We have that $\sigma\langle W_1, U\rangle \in J_j$ and $\sigma\langle W_2, U\rangle \in I_j \setminus J_j$ thus we can apply the above reasoning to both of them separately and get, using linearity:

$$\begin{aligned}
&\partial_{j+1}g_{j+1}(\mathrm{Cone}_T(f_j(\mathbb{1}_\sigma)) + g_{j+1}(\mathrm{Cone}_T(f_j(\partial_j\mathbb{1}_\sigma)))\\
&= \partial_{j+1}g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_{\sigma\langle W_1,U\rangle})) + g_{j+1}(\mathrm{Cone}_T(\partial_j\mathbb{1}_{\sigma\langle W_1,U\rangle})))\\
&\quad + \partial_{j+1}g_{j+1}(\mathrm{Cone}_T(\mathbb{1}_{\sigma\langle W_2,U\rangle})) + g_{j+1}(\mathrm{Cone}_T(\partial_j\mathbb{1}_{\sigma\langle W_2,U\rangle})))\\
&= \mathbb{1}_{(\sigma\langle W_1,U\rangle)\langle U,W_U\rangle} + 0 = \mathbb{1}_\sigma.
\end{aligned}$$

For the last equality, we used that $(\sigma\langle W_1, U\rangle)\langle U, W_U\rangle = \sigma$ since we first replace W_1 in σ by U and by applying $\langle U, W_U\rangle$ we replace U by $W_U = W_1$ and end up where we started.

For bounding the cone radius of $\mathrm{Cone}_{\tilde{T}}$ note that f_j at most doubles the size of support of a chain, while applying g_j will not increase the size of the support of a chain. Hence in the worst case, i.e. if $\sigma \in I_j$ then

$$|\mathrm{supp}(\mathrm{Cone}_{\tilde{T}}(\mathbb{1}_\sigma))| \le |\mathrm{supp}(\mathrm{Cone}_T(\mathbb{1}_{\sigma\langle W_1,U\rangle}))| + |\mathrm{supp}(\mathrm{Cone}_{\tilde{T}}(\mathbb{1}_{\sigma\langle W_2,U\rangle}))| \le 2\,\mathrm{Rad}_j\,\mathrm{Cone}_T \le 2c.$$

Hence we get the desired bound on the cone radius. □

For the D_n case, we need the following weaker and more specialized version of Theorem 6.1. In order to state it, we need the following definition.

Definition 6.11. *Let K be a field, V a K-vector space, Y a collection of subspaces of V. Let $X = \operatorname{Flag} Y$ and $U \in Y$. Then we define*

$$Y^{<U} := \{W \in Y \mid W < U\},\ X^{<U} := \operatorname{Flag} Y^{<U}$$
$$Y^{>U} := \{W \in Y \mid W > U\},\ X^{>U} := \operatorname{Flag} Y^{>U}.$$

Remark 6.12. *Note that*

$$\operatorname{lk}_X(U) = X^{<U} * X^{>U}.$$

Theorem 6.13. *Let $\mathbf{C}$ be a class of tuples $(\kappa, (V, Q, f))$ where (V, Q, f) is a thick pseudo-quadratic space and $\kappa = \operatorname{Flag} Y$ where $Y \neq \emptyset$ is set of vector subspaces of V. Let $n = \dim \kappa$, assume there exists $\ell_n \in \mathbb{N}$ and a filtration $\kappa_0 \subset \kappa_1 \subset \ldots \subset \kappa_\ell = \kappa, \ell \leq \ell_n$ satisfying:*

1. *There exists an $(n-1)$-cone function $\operatorname{Cone}_{\kappa_0}$ such that*

$$\operatorname{Rad}_j(\operatorname{Cone}_{\kappa_0}) \leq f(n), -1 \leq j \leq n-1$$

   where $f : \mathbb{R} \to \mathbb{R}$ is a function depending only on $\mathbf{C}$.

2. *Every κ_i is a full subcomplex of κ, i.e. $\kappa_i = \operatorname{Flag} Y_i$ for some $Y_i \subseteq Y$ and all vector subspaces in $Y_i \setminus Y_{i-1}$ have the same dimension.*

3. *For every $U \in Y_i \setminus Y_{i-1}$ we have that $\operatorname{lk}_{\kappa_i}(U) \cap \kappa_{i-1} = \kappa_{i-1}^{<U} * \kappa_{i-1}^{>U}$ is such that $\kappa_{i-1}^{<U}, \kappa_{i-1}^{>U} \in \mathbf{C}$ or one (or both) have again a filtration $R_0 \subset \ldots R_h, h \leq \ell_{i-1}$ which starts with an element of $\mathbf{C}$ (i.e. $R_0 \in \mathbf{C}$), such that condition 2. is satisfied and such that for every $W \in R_j(0) \setminus R_{j-1}(0)$ we have $R_{j-1}^{<W}, R_{j-1}^{>W} \in \mathbf{C}$.*

Then there exist constants $\mathcal{R}(n)$ depending only on $\mathbf{C}$ and n, and an $(n-1)$-cone function $\operatorname{Cone}_\kappa$ such that

$$\operatorname{Rad}_j(\operatorname{Cone}_\kappa) \leq \mathcal{R}(n), -1 \leq j \leq n-1.$$

Proof. The proof is analogues to the one of Theorem 6.1. The main difference is that to get a cone function for $\operatorname{lk}_{\kappa_i}(U) \cap \kappa_{i-1}$ it might happen that we cannot just use the induction hypothesis, but that we need to do another induction on the filtration of $\kappa_{i-1}^{<U}$ or $\kappa_{i-1}^{>U}$, using Theorem 3.2 in the step of the induction. This gives an even weaker and more complicated bound on the cone radius of the final cone function, thus we decided to not keep track of it. But it will still only depend on n and the class $\mathbf{C}$, not on the field K. □

We now describe the set up for the rest of this section. We follow [Abr96, Section 7] closely and also refer to that book for more details. We will define three subclasses $\mathbf{C}_1, \mathbf{C}_2, \mathbf{C}_3$ and first show that $\mathbf{C}_1, \mathbf{C}_2$ satisfy the assumptions of Theorem 6.13 to then conclude that $\mathbf{C}_D = \mathbf{C}_1 \cup \mathbf{C}_2 \cup \mathbf{C}_3$ satisfies them as well.

We fix n to be the Witt index of (V, Q, f) hence $\dim V = 2n$ and we are dealing with a D_n building, which is $n-1$-dimensional. Recall $X = \{0 < U < V \mid U \text{ totally isotropic}\}$.

Definition 6.14. *[Abr96, Definition 12] Let $U, E < V$, $\dim U < n, \dim E = n$. Assume that E is not totally isotropic. Then*

$$U @ E :\iff U^\perp \cap E \textit{ is not totally isotropic.}$$

Set

$$\mathcal{U}_n = \mathcal{U}_n(V) = \{A < V \mid \dim A = n, A \notin X\}$$
$$\mathcal{M}(A) = \{M < A \mid \dim M = n-1, M \in X\} \text{ for any } A \in \mathcal{U}_n.$$

Definition 6.15. *We define the class* $\mathbf{C}_1$ *to consist of complexes* $\operatorname{Flag} X_{\mathcal{E};\mathcal{F}}(U;V)$ *where*

- (V, Q, f) *is thick pseudo-quadratic space of Witt index* n*, in particular* $\dim V = 2n$*;*
- $U \in X$ *with* $\dim U = k \geq 2$*;*
- $\mathcal{E}$ *a finite set of subspaces of* U*, set* $e_j = |\mathcal{E}_j|, 1 \leq j \leq k-1$ *like before;*
- *Let* $\mathcal{F} = \mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots \supset \mathcal{F}_{k-1}$ *be finite subsets of* $\mathcal{U}_n(V)$ *satisfying*
    1. $U \cap F \in \mathcal{E} \cup \{0\}$ *for all* $F \in \mathcal{F}$*;*
    2. $\mathcal{F}_j^{\perp} = \mathcal{F}_j$*;*
    3. $\dim(U \cap F) \leq k - 1 - j$ *for all* $F \in \mathcal{F}_j, 1 \leq j \leq k-1$*.*
- *Assume that* $|K| \geq \sum_{j=1}^{k-1} \binom{k-2}{j-1} e_j + 2^{k-1}s$*, where* $s = |\mathcal{F}|$*.*

Set

$$X_{\mathcal{E};\mathcal{F}}(U;V) := \{0 < W < U \mid W \pitchfork_U \mathcal{E} \text{ and } W @_V \mathcal{F}_j \text{ for } \dim W = j\}.$$

Lemma 6.16. *Let* $\kappa = \operatorname{Flag} X_{\mathcal{E};\mathcal{F}}(U;V) \in \mathbf{C}_1$ *with* $k = \dim U \geq 2$*, in particular* κ *has dimension* $k-2$*. Then there exists a 1-dimensional subspace* $\ell \in X_{\mathcal{E};\mathcal{F}}(U;V)$*. We set* $Y = X_{\mathcal{E};\mathcal{F}}(U;V)$ *and define*

$$Y_0 := \{A \in Y \mid A + \ell \in Y\}.$$

Then $\kappa_0 = \operatorname{Flag} Y_0$ *has a* $k-3$ *cone function* $\operatorname{Cone}_{\kappa_0}$ *with*

$$\operatorname{Rad}_j(\operatorname{Cone}_{\kappa_0}) \leq k, \ -1 \leq j \leq k-3.$$

Proof. We proceed similar to the proof of Lemma 6.5. Set $B_0 := \{A \in Y \mid \ell \leq A\}, \beta_0 = \operatorname{Flag} B_0$. Then $\beta_0 = \{\ell\} * \operatorname{lk}_Y(\{\ell\})$ and hence it has a $k-3$- cone function with cone radius bounded by 1. Next, set $B_i = \{A \in Y_0 \mid \ell \leq A \text{ or } \dim A \leq i\}, \beta_i = \operatorname{Flag} B_i$ for $1 \leq i \leq k-1$. Then all subspaces in $B_i \setminus B_{i-1}$ have dimension i. Let $A \in B_i \setminus B_{i-1}$. Then $\operatorname{lk}_{\beta_i}(A) \cap \beta_{i-1} = \{A + \ell\} * \operatorname{Flag}\{W \in B_{i-1} \mid B \neq A + \ell, A < W \text{ or } W < A\}$ by the same argument as in Lemma 6.5. Applying Theorem 3.2 inductively, we get a $(k-3)$-cone function $\operatorname{Cone}_{\beta_{k-1}} = \operatorname{Cone}_{\kappa_0}$ with

$$\operatorname{Rad}_j(\operatorname{Cone}_{\kappa_0}) \leq k, \ -1 \leq j \leq k-3.$$

□

Lemma 6.17. *The class* $\mathbf{C}_1$ *satisfies the assumptions of Theorem 6.13. In particular, any complex in* $\mathbf{C}_1$ *of dimension* $k-2$ *has a* $(k-3)$*-cone function with cone radius bounded depending only on* k*.*

The proof of this lemma follows closely the steps and arguments in the proof of [Abr96, Lemma 33], adapting it to our setting where necessary.

Proof. Note that for any $\kappa = \operatorname{Flag} X_{\mathcal{E};\mathcal{F}}(U;V) \in \mathbf{C}_1$ by [Abr96, Lemma 33, 1.] there exists a 1-dimensional subspace $\ell \in X_{\mathcal{E};\mathcal{F}}(U;V) =: Y$. In particular, if κ is 0-dimensional, it is not empty. We will proceed to prove the existence of a desired filtration. Let $k = \dim U = \dim \kappa + 2$. For $k = 2$, this follows since $\kappa \neq \emptyset$ and we can thus define a (-1)-cone function for κ with cone radius 1. In particular, $\ell_0 = 0$. For $k \geq 3$, let $\kappa \in \mathbf{C}_1$ with $\dim \kappa \geq 1$. Fix a 1-dimensional subspace $\ell \in Y$ and set $Y_0 := \{A \in Y \mid A + \ell \in Y\}, \kappa_0 = \operatorname{Flag} Y_0$. By Lemma 6.16, there exists a $(k-3)$-cone function $\operatorname{Cone}_{\kappa_0}$ such that

$$\operatorname{Rad}_j(\operatorname{Cone}_{\kappa_0}) \leq k =: f(k), -1 \leq j \leq k-3.$$

The filtration that we will use consists of two dimension decreasing filtrations. First we define

$$\begin{aligned} Z &:= \{W \in Y \mid \ell \leq W \text{ or } W \pitchfork_U (\mathcal{F} \cap U) + \ell\} \supseteq Y_0 \\ Y_i &:= \{W \in Z \mid W \in Y_0 \text{ or } \dim W \geq k - i\}, \\ \kappa_i &:= \operatorname{Flag} Y_i, \ 1 \leq i \leq k-1. \end{aligned}$$

This satisfies assumption 2. in Theorem 6.13.

Let $W \in Y_i \setminus Y_{i-1}$. Then $Y_{i-1}^{<W} = Y_0^{<W}$ by looking at the dimension. Step 3 in the proof of [Abr96, Lemma 33] shows that $Y_0^{<W} = X_{\mathcal{E}';\mathcal{F}'}(W;V)$ for some $\mathcal{E}', \mathcal{F}'$. Thus $\kappa_{i-1}^{<W} \in \mathbf{C}_1$. On the other hand, $Y_{i-1}^{>W} = Z^{>W}$. Step 5 in [Abr96, Lemma 33] shows that while $\operatorname{Flag} Z^{>W}$ is not necessary in $\mathbf{C}_1$ it admits itself a filtration of length i satisfying the necessary properties.

The second part of the filtration of κ is defined as follows:

$$\begin{aligned} Z_i &:= \{W \in Y \mid W \in Z \text{ or } \dim W \geq k-1\}, \\ \kappa_{n+i} &:= \operatorname{Flag} Z_i, 0 \leq i \leq k-1. \end{aligned}$$

This part of the filtration clearly satisfies assumption 2. again. Let $W \in Z_i \setminus Z_{i-1}$. Then $Z_{i-1}^{>W} = Y^{>W}$ and $\operatorname{Flag} Y^{>W} \in \mathbf{C}_1$ by Step 4 in [Abr96, Lemma 33]. On the other hand, $Z_{i-1}^{<W} = Z^{<W}$ for which we again have $\operatorname{Flag} Z^{<W} \in \mathbf{C}_1$ by [Abr96].

□

Definition 6.18. *We define the class* $\mathbf{C}_2$ *to consist of complexes* $\operatorname{Flag} Z_{\mathcal{E};\mathcal{F}}(U;V)$ *where*

- (V, Q, f) *is a thick pseudo-quadratic space of Witt index* n*, in particular* $\dim V = 2n$*;*
- $U \in X$ *with* $\dim U = k \geq 2$*;*
- $\mathcal{E}$ *a finite set of subspaces of* U*, set* $e_j = |\mathcal{E}_j|, 1 \leq j \leq k-1$ *like before;*
- $\mathcal{F} \subset \mathcal{U}_n(V)$ *with* $|\mathcal{F}| = s < \infty$ *such that*
  1. $U \cap \mathcal{F} \subseteq \mathcal{E} \cup \{0\}$,
  2. $U \cap F^{\perp} = 0$ *and* $\dim(U \cap F) \leq 1$ *for all* $F \in \mathcal{F}$*.*
- *Assume* $|K| \geq \sum_{j=1}^{k-1} \binom{k-2}{j-1} e_j + 2s$*.*
- *Set*

$$Z_{\mathcal{E};\mathcal{F}}(U;V) := \{0 < W < U \mid W \pitchfork_U \mathcal{E} \text{ and } W @_V \mathcal{F}\}.$$

Lemma 6.19. *Let* $\kappa = \operatorname{Flag} Z_{\mathcal{E};\mathcal{F}}(U;V) \in \mathbf{C}_2$*,* $\dim U = k \geq 2$*, in particular* $\dim \kappa = k-2$*. Set* $Y = Z_{\mathcal{E};\mathcal{F}}(U;V)$*. Then there exists* $H < U$ *with* $\dim H = k-1$ *and* $H \in Y$*. We define*

$$Y_0 := \{B \in Y \mid B \cap H \in Y\}.$$

Then $\kappa_0 = \operatorname{Flag} Y_0$ *has a* $(k-3)$*-cone function* $\operatorname{Cone}_{\kappa_0}$ *with*

$$\operatorname{Rad}_j(\operatorname{Cone}_{X_0}) \leq k, \ -1 \leq j \leq k-3.$$

Proof. The proof is similar to the proofs of Lemma 6.5 and Lemma 6.16, the difference being that we now look at the dual version, where we fix a hyperplane instead of a line.

Set $B_0 = \{A \in Y \mid A \leq H\}, \beta_0 = \operatorname{Flag} B_0$. Then $\beta_0 = \{H\} * \operatorname{lk}_{Y_0}(\{H\})$ hence it has a $(k-3)$-cone function with cone radius bounded by 1.

Next, set $B_i = \{A \in Y_0 \mid A \leq H \text{ or } \dim A \geq k-i\}, \beta_i = \operatorname{Flag} B_i$ for $1 \leq i \leq k-1$. Then the subspaces in $B_i \setminus B_{i-1}$ are all of dimension $k-i$. Let $A \in B_i \setminus B_{i-1}$, then $\operatorname{lk}_{\beta_i}(A) \cap \beta_{i-1} = \{A \cap H\} * \operatorname{Flag}\{W \in B_{i-1} \mid W \neq A \cap H, W < A \text{ or } A < W\}$. Hence it has an $(k-4)$-cone function with cone radius bounded by 1. Thus we can apply Theorem 3.2 inductively and obtain a $(k-3)$-cone function $\operatorname{Cone}_{B_{k-1}} = \operatorname{Cone}_{\kappa_0}$ with

$$\operatorname{Rad}_j(\operatorname{Cone}_{\kappa_0}) \leq k, \ -1 \leq j \leq k-3.$$

□

Lemma 6.20. *The class* $\mathbf{C}_2$ *satisfies the assumptions of Theorem 6.13. In particular, any complex in* $\mathbf{C}_2$ *of dimension* $k-2$ *has a* $(k-3)$*-cone function with cone radius bounded depending only on* k*.*

The proof of this lemma follows closely the steps and arguments in the proof of [Abr96, Lemma 34], adapting it to our setting where necessary.

Proof. Let $\kappa = \operatorname{Flag} Z_{\mathcal{E};\mathcal{F}}(U;V) \in \mathbf{C}_2$. Then by [Abr96, Step (3), Lemma 34] there exists $H < U$ such that $\dim H = \dim U - 1 = k-1$ and $H \in Z_{\mathcal{E};\mathcal{F}}(U;V) =: Y$. In particular, $\kappa \neq \emptyset$. We proof the existence of a desired filtration by induction on k. For $k = 2$ we have that $\dim \kappa = 0$ hence the desired filtration and cone function exist trivially. Fix $k \geq 3$ and assume $\dim \kappa = k-2$. We set κ_0 as in Lemma 6.19:

$$Y_0 := \{B \in Y \mid B \cap H \in Y\}, \kappa_0 = \operatorname{Flag} Y_0.$$

Thus Lemma 6.19 shows the existence of the desired cone function for κ_0 with $f(n) = n$. We define the filtration in the following way.

$$\begin{aligned} Y_i &:= \{W \in Y \mid W \in Y_0 \text{ or } \dim W \leq i\}, \\ \kappa_i &:= \operatorname{Flag} Y_i, \ 1 \leq i \leq k-1. \end{aligned}$$

This clearly satisfies assumption 2. of Theorem 6.13. To see that it satisfies assumption 3., let $W \in Y_i \setminus Y_{i-1}$. Then $Y_{i-1}^{<W} = Y^{<W} = \{0 < A < W \mid A \pitchfork_W \mathcal{E} \cap W\} = Z_{\mathcal{E} \cap W;\emptyset}(W;V)$ and hence $\operatorname{Flag} Y^{<W} \in \mathbf{C}_2$. On the other hand, $Y_{i-1}^{>W} = Y_0^{>W}$ can again be described in such a way that $\operatorname{Flag} Y_0^{>W} \in \mathbf{C}_2$, see Step (5) in [Abr96, Lemma 34]. Since $\kappa_{k-1} = \kappa$, the filtration is indeed as desired. □

Next, we will define a third class, which will be the one containing the relevant opposition complexes, see also Remark 6.24.

Definition 6.21. *We define the class* $\mathbf{C}_3$ *to consist of complexes of the form* $\operatorname{Flag} Y_{\mathcal{E}}(V)$ *where*

- (V, Q, f) *is a thick pseudo-quadratic space of Witt index* n*, in particular* $\dim V = 2n$*;*
- $\mathcal{E} = \mathcal{E}^{\perp}$ *is a finite set of subspaces of* V*, set* $\mathcal{E}_j = \{E \in \mathcal{E} : \dim E = j\}, e_j = |\mathcal{E}_j|$*;*
- $|K| \geq 2\sum_{j=1}^{2n-1} \binom{2n-2}{j-1} e_j$*;*
- $\widehat{\mathcal{E}}_n := \mathcal{E}_n \cap \mathcal{U}_n(V)$*;*
- $X = \{0 < U < V \mid U \text{ is totally isotropic}\}$*.*

Set

$$Y_{\mathcal{E}}(V) := \{U \in X \mid U \tilde{\pitchfork} \mathcal{E} \text{ and if } \dim U < n \text{ then } U @ \hat{\mathcal{E}}_n\}.$$

Lemma 6.22. *Let* $\kappa = \operatorname{Flag} Y_{\mathcal{E}}(V) \in \mathbf{C}_3$ *with* $\dim V = 2n$*. Then there exists a 1-dimensional subspace* $\ell \in Y = Y_{\mathcal{E}}(V)$*. Set*

$$\begin{aligned} \mathcal{F} &:= \mathcal{E} \cup \left(\mathcal{E} \cap \ell^{\perp}\right) \cup (\mathcal{E} + \ell) \cup \left(\left(\mathcal{E} \cap \ell^{\perp}\right) + \ell\right) \text{ and} \\ \widehat{\mathcal{F}}_n &:= \mathcal{F}_n \cap \mathcal{U}_n(V). \end{aligned}$$

Consider the following conditions

1. $\ell \leq U, U \tilde{\pitchfork} \mathcal{E}$ *and if* $\dim U < n$ *then* $U @ \widehat{\mathcal{E}}_n$*;*
2. $\ell \not\leq U \leq \ell^{\perp}$ *and* $\begin{cases} U \pitchfork \mathcal{E}, U @ \widehat{\mathcal{E}}_n & \text{if } \dim U = n-1, \\ U \pitchfork \mathcal{E} \cup (\mathcal{E} + \ell), U @ \widehat{\mathcal{F}}_n & \text{if } \dim U \leq n-2; \end{cases}$
3. $U \not\leq \ell^{\perp}, \dim U > 1, U \widetilde{\pitchfork} \mathcal{F}$ *and if* $\dim U < n$ *then* $U @ \widehat{\mathcal{F}}_n$*.*

We define $Y_0 := \{U \in X \mid U \text{ satisfies 1.,2., or 3. }\}$*. Then* $\kappa_0 = \operatorname{Flag} Y_0$ *has an* $(n-2)$*-cone function* $\operatorname{Cone}_{\kappa_0}$ *with*

$$\operatorname{Rad}_j(\operatorname{Cone}_{\kappa_0}) \leq 2n + 1, \ -1 \leq j \leq n-2.$$

Proof. Let $\kappa = \operatorname{Flag} Y_{\mathcal{E}}(V) \in \mathbf{C}_3$ with $\dim V = 2n$. By Step 1. [Abr96, Proposition 14] there exists a 1-dimensional subspace $\ell \in Y_0$. Let Y_0 be defined as above. Recall that $X = \{0 < U < V \mid U \text{is totally isotropic subspace}\}$. We define a filtration for Y_0 and we want to apply Theorem 3.2 inductively. We will use the fact stated in Step 3 of [Abr96, Proposition 14], that for every $U \in Y_0$, we have that $U \cap \ell^{\perp} \in Y_0$ and $(U \cap \ell^{\perp}) + \ell \in Y_0$.

Set

$$\begin{aligned} A_0 &:= \{U \in X \mid U \text{ satisfies 1.}\} \\ A_i &:= \{U \in X \mid U \text{ satisfies 1. or } (\dim U \leq i \text{ and } U \text{ satisfies 2.})\}, \ 1 \leq i \leq n; \\ \alpha_i &:= \operatorname{Flag} A_i, 0 \leq i \leq n. \end{aligned}$$

Then $A_0 = \{\ell\} * \operatorname{Flag}\{U \in A_0 \mid U \neq \ell\}$ and hence has a cone function with cone radius bounded by 1. Clearly elements in $A_i \setminus A_{i-1}$ have the same dimension an are hence not adjacent. Let $U \in A_i \setminus A_{i-1}$. Then

$$\operatorname{lk}_{\alpha_i}(U) \cap \alpha_{i-1} = \operatorname{Flag}\{W \in X \mid (W \in A_0 \text{ and } U < W) \text{ or } (W \in A_{i-1} \text{ and } W < U)\}.$$

We want to show that each subspace in $\{W \in X \mid (W \in A_0 \text{ and } U < W) \text{ or } (W \in A_{i-1} \text{ and } W < U)\}$ is either contained in or contains $U + \ell$ (note that we have $U \cap \ell^\perp + \ell = U + \ell \in A_0 \subseteq A_{i-1}$). If $W \in A_0$ and $U < W$ then $\ell \leq W$ and hence $U + \ell \leq W$. If $W \in A_{i-1}$ and $W < U$ then $W < U + \ell$ as needed. The second part of the filtration is defined as follows. We set

$$\begin{aligned} B_i &:= \{U \in X \mid U \in A_n \text{ or } (U \text{ satisfies 3. and } \dim U \geq n - i + 1\}, \\ \beta_i &:= \operatorname{Flag} B_i, 0 \leq i \leq n. \end{aligned}$$

Then $A_n = B_0$. Let $U \in B_i \setminus B_{i-1}$. We want to show that $\operatorname{lk}_{\beta_i}(U) \cap \beta_{i-1}$ can be written as the join of $\{U \cap \ell^\perp\}$ and $\operatorname{Flag}\{W \in B_{i-1} \mid W \neq U \cap \ell^\perp, W < U \text{ or } U < W\}$. First, note that $U \cap \ell^\perp \in Y_0$ and hence $U \cap \ell^\perp \in A_n \subseteq B_{i-1}$. Let $W \in B_{i-1} \setminus \{U \cap \ell^\perp\}$. If $W < U$ then $\dim W < n - i$, hence $W \in A_n \setminus A_0$, in particular $W \leq \ell^\perp$. Hence $W = W \cap \ell^\perp \leq U \cap \ell^\perp$. If $U < W$ then $U \cap \ell^\perp \leq U \leq W$. Thus we get a filtration of κ_0 of length $2n$. By applying Theorem 3.2 at each step, we get an $n-2$ cone function $\operatorname{Cone}_{\kappa_0}$ with

$$\operatorname{Rad}_j(\operatorname{Cone}_{\kappa_0}) \leq 2n + 1, -1 \leq j \leq n - 2.$$

□

We now have all the necessary ingredients to show that the relevant class of simplicial complexes satisfies the conditions of Theorem 6.13. The prove will be similar to the previous proofs of similar results.

Theorem 6.23. *The class $\mathbf{C}_D = \mathbf{C}_1 \cup \mathbf{C}_2 \cup \mathbf{C}_3$ satisfies the conditions of Theorem 6.13.*

Proof. Let $\kappa = \operatorname{Flag} Y \in \mathbf{C}_D$. If $\kappa \in \mathbf{C}_1 \cup \mathbf{C}_2$ then the result follows from Lemma 6.17 and Lemma 6.20.

Now assume $\kappa = \operatorname{Flag} Y_{\mathcal{E}}(V) \in \mathbf{C}_3$. In this case the proof follows closely the proof of [Abr96, Proposition 14]. First of all, notice that we again have a 1-dimensional subspace $\ell \in Y_{\mathcal{E}}(V) =: Y$ (see Step 1 [Abr96, Proposition 14]). Let $n = \frac{1}{2}\dim V = \dim \kappa + 1$. We will again prove the existence of a filtration by induction on n, but this time starting with $n = 2$.

Let $n = 2$, hence $\dim \kappa = 1$. Step 2. in [Abr96, Proposition 14] shows that each vertex $U \in Y$ can be connected to ℓ by a path of length at most 5. Thus we define a 0-cone function of κ as follows:

$$\begin{aligned} \operatorname{Cone}_\kappa(\emptyset) &= \mathbb{1}_\ell \\ \operatorname{Cone}_\kappa(\mathbb{1}_{[U]}) &= \sum_{i=0}^{k} \mathbb{1}_{[U_i, U_{i+1}]}, \end{aligned}$$

where $U = U_0, \ldots, U_k = \ell$ is the path from U to ℓ for $U \in Y \setminus \{\ell\}$. In particular, $k \leq 4$. By Example 2.5, this is indeed a cone function. We can read off the cone radius from the explicit description and get

$$\operatorname{Rad}_{-1}(\operatorname{Cone}_\kappa) = 1, \operatorname{Rad}_0(\operatorname{Cone}_\kappa) \leq 5.$$

Now assume $n \geq 3$. We define $\kappa_0 = \operatorname{Flag} Y_0$ as in Lemma 6.22. In particular, we know that there exists an $(n-2)$-cone function $\operatorname{Cone}_{\kappa_0}$ with $\operatorname{Rad}_j(\operatorname{Cone}_{\kappa_0}) \leq 2n+1 = f(n)$ for all $-1 \leq j \leq n-2$.

The filtration will contain two parts, one dimension increasing and one dimension decreasing filtration. To define the first, we define what we call condition 4. in the following way, using the notation from Lemma 6.22.

$$U \not\leq \ell^{\perp} \text{ and } \begin{cases} U \tilde{\pitchfork} \mathcal{F} & \text{if } \dim U = n \\ U \pitchfork \mathcal{F} \text{ and } U @ \widehat{\mathcal{F}}_n & \text{if } \dim U = n-1 \\ U \pitchfork \mathcal{E} \cup (\mathcal{E} + \ell) \text{ and } U @ \widehat{\mathcal{F}}_n & \text{if } \dim U = n-2. \end{cases}$$

Set

$$Z := \{U \in X \mid U \text{ satisfies } 1., 2., \text{ or } 4.\}$$
$$Y_i := \{U \in Z \mid U \in Y_0 \text{ or } \dim U \leq i\}, 1 \leq i \leq n-2.$$

Note that $Y_0 \subseteq Z$ since if $\dim U < n$ then $U \tilde{\pitchfork} \mathcal{E} \iff U \pitchfork \mathcal{E}$.

Let $U \in Y_i \setminus Y_{i-1}$. We have that $U \not\leq \ell^{\perp}$ and $\dim U = i \leq n-2$. We get $Y_{i-1}^{<U} = Z^{<U}$. Since $\ell \not\leq U$ (otherwise $U \leq U^{\perp} \leq \ell^{\perp}$), we can apply Step 5 of the proof of [Abr96, Proposition 14] to get that $\operatorname{Flag} Z^{<U} \in \mathbf{C}_2$.

On the other hand, $Y_{i-1}^{>U} = Y_0^{>U}$ and we have $\operatorname{Flag} Y_0^{>U} \in \mathbf{C}_3$ by [Abr96].

Next, we treat $U \in Y$ with $\dim U = n-1$. To cover these elements, we define

$$Y_{n-1} := Z \cup \{U \in Y \mid \dim U = n-1 \text{ and } Y_0^{>U} \neq \emptyset\}.$$

Let $U \in Y_{n-1} \setminus Y_{n-2}$. Then $Y_{n-2}^{<U} = Z^{<U}$ and $\operatorname{Flag} Z^{<U} \in \mathbf{C}_2$. Furthermore, we have $Y_{n-2}^{>U} = Y_0^{>U} \neq \emptyset$. Note that $\dim \operatorname{Flag} Y_0^{>U} = 0$ since $Y_0^{>U}$ only contains subspaces of dimension n. Hence it trivially has a (-1)-cone function with cone radius 1.

The dimension increasing filtration is defined by setting

$$Y_i := \{U \in Y \mid U \in Y_{n-1} \text{ or } \dim U \geq 2n-i\}, n \leq i \leq 2n-1.$$

Let $U \in Y_i \setminus Y_{i-1}$ and set $k = 2n - i = \dim U$. We distinguish the cases $k = n$ and $k < n$.

If $k = n$, then $Y_{n-1}^{>U} = \emptyset$ since there is no totally isotropic subspace of dimension $> n$. This is fine for our sake, since $\emptyset * A = A$ for an arbitrary simplicial complex A. On the other hand, $Y_{n-1}^{<U}$ has again a filtration starting from an element in $\mathbf{C}_1$. To describe the filtration, we set $\mathcal{E}' := (\mathcal{E} \cap U) \cup ((\mathcal{E} + \ell) \cap U), \mathcal{F}' := \mathcal{F} \cap U, \mathcal{H} := \mathcal{H}_1 := \cdots := \mathcal{H}_{n-2} = \widehat{\mathcal{F}_n}$ and $\mathcal{H}_{n-1} := \{F \in \widehat{\mathcal{F}}_n \mid U \cap F = 0\}$. Furthermore, we set

$$S := X_{\mathcal{E}';\mathcal{H}}(U;V) \cap \{0 < W < U \mid F' \not\leq W \text{ for all } 0 \neq F' \in \mathcal{F}'\}.$$

By Step (20) in [Abr96, Proposition 14], we know that $S \leq Y_{n-1}^{<U}$ and given a cone function of S we get a cone function of $Y_{n-1}^{<U}$ by one application of Theorem 3.2. To see that S has a cone function, we note that if $W \pitchfork_U F'$ then this implies that $F' \not\leq W$. Thus we set $\mathcal{E}'' = \mathcal{E}' \cup \mathcal{F}'$ and get $X_{\mathcal{E}'',\mathcal{H}}(U;V) \subseteq S$. We set

$$A_0 := X_{\mathcal{E}'',\mathcal{H}}(U;V), A_i = \{W \in S \mid W \in A_0 \text{ or } \dim W \leq i\}, 1 \leq i \leq n-1.$$

We follow the by now standard procedure to check that this is a valid filtration. Let $W \in A_i \setminus A_{i-1}$. Then $A_{i-1}^{>W} = A_0^{>W} = X_{\mathcal{E}'',\mathcal{H}}(U;V)^{>W}$ and by Step 4. in [Abr96, Lemma 33] Flag $X_{\mathcal{E}'',\mathcal{H}}(U;V)^{>W} \in \mathbf{C}_1$. On the other hand, $A_{i-1}^{<W} = S^{<W}$. Note that if $W \in S$ and $W' \in X_{\mathcal{E}';\mathcal{H}}(U;V)$ such that $W' \leq W$ then $W' \in S$ since if $F' \not\leq W$ then $F' \not\leq W' \leq W$. Thus

$$S^{<W} = X_{\mathcal{E}';\mathcal{H}}(U;V)^{<W} = \{0 < W' < W \mid W' \cap \mathcal{E}', W'@\mathcal{H}_j \text{ for } \dim W' = j\} = X_{\mathcal{E}' \cap W,\mathcal{H}}(W;V)$$

and Flag $X_{\mathcal{E}' \cap W,\mathcal{H}}(W;V) \in \mathbf{C}_1$. Hence we got a filtration

$$\operatorname{Flag} A_0 \subseteq \ldots \operatorname{Flag} A_{n-1} = \operatorname{Flag} S \subseteq \operatorname{Flag} Y_{n-1}^{<U}$$

of length $n = \dim U$ which satisfies the assumptions.

If $\dim U < n$, then $Y_{i-1}^{<U} = Z^{<U}$ and we again have Flag $Z^{<U} \in \mathbf{C}_3$. Additionally, we have $Y_{i-1}^{>U} = Y^{>U}$ for which Abramenko shows that Flag $Y^{>U} \in \mathbf{C}_3$.

Hence all the condition of Theorem 6.13 are satisfied. □

The following remark from [Abr96] gives the connection between $X_{\mathcal{E}}(V)$ and $Y_{\mathcal{E}}(V)$.

Remark 6.24. *Let $\tau = \{E_1, \ldots E_r\}$ be a simplex of $\tilde{\Delta} = \operatorname{Orifl} \tilde{X}$ and set $\mathcal{E} = \mathcal{E}(a) = \{E_i, E_i^{\perp} \mid 1 \leq i \leq r\}$. Then $\mathcal{E} \cap \mathcal{U}_n = \emptyset$, $Y_{\mathcal{E}}(V) = X_{\mathcal{E}}(V)$, $e_n \leq 2, e_{n-1} = e_{n+1} = 0$ and $e_i \leq 1$ for all other $1 \leq i \leq 2n-1$.*

Corollary 6.25. *Let $\tilde{\Delta}$ be a building of type D_n over the field K, and $a \in \tilde{\Delta}$ be a simplex. If $|K| \geq 2^{2n-1}$, then $\tilde{\Delta}^0(a)$ has an $(n-2)$-cone function with $\operatorname{Rad}_j(\operatorname{Cone}_{\tilde{\Delta}^0(a)}) \leq 2\mathcal{R}(n-1), -1 \leq j \leq n-2$, where $\mathcal{R}(n)$ does not depend on K.*

Proof. By Theorem 6.23, we have that $Y_{\mathcal{E}(a)}(V)$ has cone function with radius $\leq \mathcal{R}(n)$. By Remark 6.24, we have $X_{\mathcal{E}(a)}(V) = Y_{\mathcal{E}(a)}(V)$. Thus 6.10 yields a cone function for $\tilde{\Delta}^0(a) = \tilde{T}_{\mathcal{E}(a)}(V)$ with cone radius bounded by $2\mathcal{R}(n)$. □

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